18.06, Fall 2004, Problem Set 4 Solutions

1. (13 pts.)

   (a)

$$A = \begin{bmatrix} 0 & 0 & 2 & -2 & 1 & 2 \\ 3 & 6 & 0 & 9 & 0 & 3 \\ 1 & 2 & 0 & 3 & 1 & 3 \\ -1 & -2 & 2 & -5 & 0 & -1 \end{bmatrix}.$$

Permuting rows 1 and 2, we get:

$$\begin{bmatrix} 3 & 6 & 0 & 9 & 0 & 3 \\ 0 & 0 & 2 & -2 & 1 & 2 \\ 1 & 2 & 0 & 3 & 1 & 3 \\ -1 & -2 & 2 & -5 & 0 & -1 \end{bmatrix}.$$

Now we can eliminate entries $(3, 1)$ and $(4, 1)$ to get:

$$\begin{bmatrix} 3 & 6 & 0 & 9 & 0 & 3 \\ 0 & 0 & 2 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & -2 & 0 & 0 \end{bmatrix}.$$

The second pivot is now element $(2, 3)$, and this pivot can be used to eliminate element $(4, 3)$:

$$\begin{bmatrix} 3 & 6 & 0 & 9 & 0 & 3 \\ 0 & 0 & 2 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{bmatrix}.$$

The next pivot is element $(3, 5)$, and it allows to eliminate element $(4, 5)$:

$$\begin{bmatrix} 3 & 6 & 0 & 9 & 0 & 3 \\ 0 & 0 & 2 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix now is in echelon form. To get the reduced row echelon form, we first scale row 1 by 1/3 and row 2 by 1/2:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We still need to eliminate entry $(2, 5)$ (as x_5 is a pivot variable) and this is done by subtracting 1/2 of row 3 from row 2:

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and this is the reduced row echelon form.

(b) The rank of A is 3 since we found 3 pivot variables: x_1, x_3 and x_5.

(c) If we take $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ and we redo the eliminations on the augmented matrix $[A|b]$, we get that $Ax = b$ is equivalent to $Ex = d$ where $d = \begin{bmatrix} b_2/3 \\ b_1/2 - b_3/2 + b_2/6 \\ b_3 - b_2/3 \\ b_4 - b_1 + b_3 \end{bmatrix}$. If we take b such that $b_4 - b_1 + b_3 \neq 0$ then $Ax = b$ has no solution.

(d) When doing the elimination with $b = \begin{bmatrix} 22 \\ 24 \\ 16 \\ 6 \end{bmatrix}$, we get (see previous subquestion) $d = \begin{bmatrix} 8 \\ 7 \\ 8 \\ 0 \end{bmatrix}$. Thus a particular solution is

$$x_p = \begin{bmatrix} 8 \\ 0 \\ 7 \\ 0 \\ 8 \\ 0 \end{bmatrix}.$$

To get all solutions, we need to add linear combinations of the special solutions of the nullspace. We have a special solution for each free variable x_2, x_4 and x_6. All solutions to $Ax = b$ are thus given by:

$$\begin{bmatrix} 8 \\ 0 \\ 7 \\ 0 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} = + \begin{bmatrix} 8 - 2x_2 - 3x_4 - x_6 \\ x_2 \\ 7 + x_4 \\ x_4 \\ 8 - 2x_6 \\ x_6 \end{bmatrix}.$$

(e) No, since the nullspace contains non-zero vectors.

(f)

$$A^T A = \begin{bmatrix} 11 & 22 & -2 & 35 & 1 & 13 \\ 22 & 44 & -4 & 70 & 2 & 26 \\ -2 & -4 & 8 & -14 & 2 & 2 \\ 35 & 70 & -14 & 119 & 1 & 37 \\ 1 & 2 & 2 & 1 & 2 & 5 \\ 13 & 26 & 2 & 37 & 5 & 23 \end{bmatrix}.$$

(g) The rank of A^TA is also 3. Indeed let us prove that the rank of A^TA is always equal to the rank of A (without doing any eliminations).

To see this, we first show that $N(A) = N(A^TA)$. It is clear that any x with $Ax = 0$ satisfies $A^TAx = 0$. The converse is also true: If $A^TAx = 0$, observe that for $w = Ax$ we have that $w \in N(A^T)$ and $w = C(A)$ which implies that $w = 0$ as $N(A^T) \cap C(A) = \{0\}$. In other words $A^TAx = 0$ implies that $Ax = 0$. The fact that $N(A) = N(A^TA)$ now implies that the dimensions of these subspaces are the same and thus we have $rank(A) = rank(A^TA)$.

2. (6 pts.) Consider the space F spanned by the 4 vectors $v_1 = (4, 2, 4, 2)$, $v_2 = (-1, 4, 5, 10)$, $v_3 = (-5, 2, 1, 8)$ and $v_4 = (6, 6, 10, 10)$.

(a) The v_i's are not linearly independent. Indeed, if you consider the matrix

$$A = \begin{bmatrix} 4 & -1 & -5 & 6 \\ 2 & 4 & 2 & 6 \\ 4 & 5 & 1 & 10 \\ 2 & 10 & 8 & 10 \end{bmatrix},$$

and do eliminations, we'll get only two pivots. The matrix A would need to have a nullspace of dimension 0 for the vectors to be linearly independent.

(b) v_1 and v_2 forms a basis of F. Any two of the v_i's would work here as none of them is a multiple of another.

(c) The dimension of F is 2 as we have two pivots.

(d) $v_1 + 2v_2 + 3v_3$, $v_1 - v_2$ and v_4 cannot be linearly independent since 3 vectors of a subspace of dimension 2 are never linearly independent.

3. (5 pts.) Consider the subspace F of all 3×3 *symmetric* matrices with zeroes on the diagonal.

(a) Consider the 3 matrices:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

A linear combination of these matrices gives the matrix:

$$\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}.$$

To get the 0 matrix, we must have $a = b = c = 0$ implying that the 3 matrices are linearly independent. Furthermore we can get any symmetric matrix with zeroes on the diagonal by choosing a, b and c appropriately, and thus these 3 vectors span the subspace. Hence they form a basis.

(b) We'll need $1 + 2 + \cdots + n - 1$ matrices in the basis, for a total of $\frac{n(n-1)}{2}$.

4. (4 pts.) Suppose we couldn't find an index l. This means that $v_1, v_2, \cdots, v_{k-1}, v_k, v_l$ are linearly dependent for every $l = k+1, \cdots, n$. Since $v_1, \cdots, v_k$ are linearly independent, it means that v_l linearly depends on $v_1, \cdots, v_k$ for $l > k$. This implies that any vector which is a linear combination of all the v_i's can be expressed as a linear combination of just $v_1, \cdots, v_k$. In other words, $v_1, \cdots, v_k$ form a basis of $C(A)$ and this contradicts the fact that the rank (and thus the dimension of $C(A)$) is greater than k.

5. (12 pts.) Exercise 14 of section 3.6 on page 181. $A = BC$ where B is invertible (since it is lower triangular with nonzeroes on the diagonal).

   - $N(A)$. The nullspace $N(A)$ is equal to $N(C)$ (since B is invertible: $BCx = 0$ if and only if $Cx = 0$). As C is in echelon form and x_4 is a free variable, we can just take that special solution as the only vector in the basis of $N(C) = N(A)$:

   $$\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

   - $R(A)$. Similarly $R(A) = R(C)$ (from $y = A^T u = C^T B^T u = C^T (B^T u)$ and B^T being invertible). We can just take all 3 row vectors of C as basis:

   $$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

   Thus the rank of A is 3.

   - $C(A)$. As the rank of A and thus the dimension of $C(A)$ is 3, we have that $C(A)$ is all of R^3. Thus we can take any basis of R^3, say the 3 unit vectors.
   - $N(A^T)$. As $dim(C(A)) + dim(N(A^T)) = 3$, we have that $dim(N(A^T)) = 0$ and thus a basis of $N(A^T)$ contains 0 vectors (not the 0 vector).