18.06 Fall 2003 Quiz 1 October 1, 2003

Your name is:

Please circle your recitation:

1. M2 S. Harvey
2. M2 D. Ingerman
3. M3 S. Harvey
4. T10 B. Sutton
5. T10 C. Taylor
6. T11 K. Cheung
7. T11 N. Ganter
8. T12 N. Ganter
9. T12 S. Francisco
10. T1 K. Cheung
11. T1 B. Tenner
12. T2 K. Cheung

Grading:

Question	Points	Maximum
Name + rec		5
1		25
2		15
3		5
4		35
5		15
Extra credit:		(10)
Total:		100

Remarks:
Do all your work on these pages.
No calculators or notes.
Putting your name and recitation name correctly is worth 5 points.
The exam is worth a total of 100 points.

1. a) (15 points) Find an LU-decomposition of the 3×3 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Solution:

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$U = E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} L &= (E_{32}E_{31}E_{21})^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \end{aligned}$$

Therefore we have,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

b) (10 points) Solve $Ax = b$ where

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution:

From 1(a) we have $A = LU$. Let $c = Ux$ and solve for $Lc = b$ using back substitution to get

$$c = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Now, solve for $Ux = c$ using back substitution to get

$$x = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}.$$

2. (15 points) Let A be an unknown 3×3 matrix, and let

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the augmented matrix $B = [A \mid P\,]$. After performing row operations on B we get the following matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -3 & -4 \\ 0 & 1 & 0 & -1 & 2 & 2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right].$$

What is A^{-1}?

Solution:

By performing 2 more row operations on B we get the following augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & -3 \\ 0 & 1 & 0 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] = \left[\, I \,\middle|\, A^{-1}P \right].$$

Since $P^{-1} = P$, we have

$$\begin{aligned} A^{-1} &= \begin{bmatrix} 2 & -3 & -3 \\ -1 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 & -3 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

3. (5 points) Find a matrix A such that

$$A \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x - y \\ x + y + 2w \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

4. All of the questions below refer to the following matrix A

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

a) (5 points) What is the rank of A?

Solution:
The rank of A is equal to the number of pivots which is 2.

b) (5 points) Do all pairs of columns span the column space, $C(A)$, of A? If yes, explain. If no, give a pair of columns that do not span the column space.

Solution:
No! The column space of A is all of $\mathbb{R}^2$. However, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are linearly dependent and hence only span a one-dimensional subspace of $\mathbb{R}^2$.

c) (10 points) Find a basis for the nullspace $N(A)$ of A.

Solution:
Let $x_2 = 1$ and $x_4 = 0$. We solve for the pivot variables: $x_1 = -2$ and $x_3 = 0$.

Let $x_2 = 0$ and $x_4 = 1$. We solve for the pivot variables: $x_1 = -1$ and $x_3 = -2$.

A basis for the nullspace is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

d) (5 points) Does there exist a vector $b \in R^2$ such that $Ax = b$ has no solution?

Solution:

No! One possible solution to $Ax = b$ is $x = \begin{bmatrix} b_1 \\ 0 \\ b_2 \\ 0 \end{bmatrix}$.

e) (10 points) Find all solutions of

$$Ax = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Express your solution in the form

$$x = x_{particular} + c_1 x_1 + c_2 x_2$$

where x_1, x_2 are special solutions.

Solution:

$$x = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

5. a) (6 points) How many 3×3 permutation matrices are there (including I)?

Solution: 3!=6

b) (9 points) Is there a 3×3 permutation matrix P, besides $P = I$, such that $P^3 = I$? If yes, give one such P. If no, explain why.

Solution: Yes,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

6. **Extra Credit (10 points)** The matrix in question 1 is a Pascal matrix. Find an LU-decomposition of the 6×6 Pascal matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 10 & 15 & 21 \\ 1 & 4 & 10 & 20 & 35 & 56 \\ 1 & 5 & 15 & 35 & 70 & 126 \\ 1 & 6 & 21 & 56 & 126 & 252 \end{bmatrix}$$

Note: you don't need to write the entire matrix again, just explain how to get the LU-decomposition.

Solution:

Let

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $L = U^T$ then $A = LU$.