Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
by Chiang C. Mei, 2004

Refs:
Pedlofsy, *Geophysical Fluid Dynamics*
Proudman: *Dynamical Oceanogrphy*
LaBlond and Mysak *Waves in the Ocean*
Gill : *Atmosphere-Ocean Dynamics.*

7.7 Free waves near a coast in a sea of constant depth

7-7KPwave.tex May 3, 2004

7.7.1 Governing equations

Let x, y be the horizontal coordinates where x is not necessarily from west to east. Recall the governing equations for a sea of constant depth H_0,

$$\frac{\partial \eta}{\partial t} + H_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0. \tag{7.7.1}$$

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{7.7.2}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{7.7.3}$$

Let us derive relations between each velocity component and the surface elevation. Differentiating (7.7.2),

$$\frac{\partial^2 u}{\partial t^2} - f\frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial t^2} - f\left[-fu - g\frac{\partial \eta}{\partial y} \right] = -g\frac{\partial^2 \eta}{\partial x \partial t} \tag{7.7.4}$$

Therefore,

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = -g\left(\frac{\partial^2 \eta}{\partial x \partial t} + f\frac{\partial \eta}{\partial y} \right) \tag{7.7.5}$$

Similarly,

$$\frac{\partial^2 v}{\partial t^2} + f^2 v = -g\left(\frac{\partial^2 \eta}{\partial y \partial t} - f\frac{\partial \eta}{\partial x} \right). \tag{7.7.6}$$

These relations are useful for specifying boundary conditions.

Let us eliminate the velocity components to get a single equation for η. From (7.7.5),

$$\left(\frac{\partial^2}{\partial t^2} + f^2 \right) \left(\frac{\partial u}{\partial x} \right) = -g\left(\frac{\partial^2 \eta}{\partial x^2 \partial t} + f\frac{\partial^2 \eta}{\partial x \partial y} \right)$$

and from (7.7.6),

$$\left(\frac{\partial^2}{\partial t^2}+f^2\right)\left(\frac{\partial v}{\partial y}\right)=-g\left(\frac{\partial^2\eta}{\partial y^2\partial t}-f\frac{\partial^2\eta}{\partial x\partial y}\right)$$

Using (7.7.1), we get

$$-\frac{1}{H_0}\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2}+f^2\right)\eta=-g\frac{\partial}{\partial t}\nabla^2\eta$$

or

$$\frac{\partial}{\partial t}\left\{\left(\frac{\partial^2}{\partial t^2}+f^2\right)\eta-C_0^2\nabla^2\eta\right\}=0 \tag{7.7.7}$$

This is the Klein-Gordon equation, where

$$C_0=\sqrt{gH_0}. \tag{7.7.8}$$

7.7.2 Waves in a long channel

Consider a channel of width L, $-\infty<x<\infty, 0<y<L$. Allowing no flux on the side walls: $v=0, y=0,L$, we have, therefore, the boundary conditions,

$$\frac{\partial^2\eta}{\partial y\partial t}-f\frac{\partial\eta}{\partial x}=0 \qquad y=0,L \tag{7.7.9}$$

Consider propagating waves. Let

$$\eta=\Re\left\{\overline{\eta}(y)e^{i(kx-\sigma t)}\right\} \tag{7.7.10}$$

We get from (7.7.7),

$$\frac{d^2\overline{\eta}}{dy^2}+\left[\frac{\sigma^2-f^2}{C_0^2}-k^2\right]\overline{\eta}=0 \qquad 0<y<L \tag{7.7.11}$$

and from (7.7.9),

$$\frac{d\overline{\eta}}{dy}+\frac{fk}{\sigma}\overline{\eta}=0 \qquad y=0,L. \tag{7.7.12}$$

The general solution is

$$\overline{\eta}=A\sin\alpha y+B\cos\alpha y, \tag{7.7.13}$$

where A and B are constants and

$$\alpha^2=\frac{\sigma^2-f^2}{C_0^2}-k^2 \tag{7.7.14}$$

Apply the boundary condition on $y=0$:

$$\alpha A+\frac{fk}{\sigma}B=0$$

and on $y = L$:

$$A\left(\alpha \cos \alpha L + \frac{fk}{\sigma} \sin \alpha L\right) + B\left(\frac{fk}{\sigma} \cos \alpha L - \alpha \sin \alpha L\right) = 0.$$

For nontrivial A and B, we must require

$$\begin{vmatrix} \alpha & fk/\sigma \\ \alpha \cos \alpha L + (fk/\sigma) \sin \alpha L & (fk/\sigma) \cos \alpha L - \alpha \sin \alpha L \end{vmatrix} = 0$$

This gives the eigenvalue equation,

$$\left(\sigma^2 - f^2\right)\left(\sigma^2 - C_0^2 k^2\right) \sin \alpha L = 0. \tag{7.7.15}$$

there are three possibilities: $\sigma = \pm f, \sigma = \pm k C_0$ and $\alpha = n\pi/L$.

7.7.3 Inertial oscillations, $\sigma^2 = f^2$

It suffices to consider $\sigma = f$. From (7.7.14),

$$\alpha^2 = -k^2 \tag{7.7.16}$$

so that

$$\frac{d^2 \overline{\eta}}{dy^2} - k^2 \overline{\eta} = 0 \qquad 0 < y < L$$

Therefore,

$$\overline{\eta} = A e^{-ky} + B e^{ky}$$

From the boundary conditions

$$\frac{d\overline{\eta}}{dy} + k\overline{\eta} = 0 \quad \text{at} \quad y = 0, L.$$

which are automatically satisfied by

$$A e^{-ky} \tag{7.7.17}$$

for any k. It easy to show that Be^{ky} cannot satisfy both boundary conditions; we must take $B = 0$. Therefore,

$$\overline{\eta} = A e^{-ky}. \tag{7.7.18}$$

Let us take a closer look at the velocity v. By eliminating u from (7.7.1) and (7.7.2), we get

$$-\frac{1}{H}\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 v}{\partial y \partial t} = f\frac{\partial v}{\partial x} - g\frac{\partial^2 \eta}{\partial x^2}$$

Using

$$\eta = A e^{-ky} e^{ikx - ift}$$

we get

$$\frac{\partial^2 v}{\partial y \partial t} + f\frac{\partial v}{\partial x} = gAe^{-ky}e^{ikx-ift}\left(-k^2 + \frac{f^2}{C^2}\right)$$

Assume the solution

$$v = Ve^{-ky}e^{ikx-ift}$$

it follows that

$$2ikV = gA\left(-k^2 + \frac{f^2}{C^2}\right)$$

Hence the boundary conditions at $y = 0, L$ dictates that

$$V = 0, \quad \text{and} \quad k^2 = \frac{f^2}{C^2} \tag{7.7.19}$$

or

$$kR = 1 \tag{7.7.20}$$

The solution is therefore

$$v = 0, \quad \eta = gAe^{-y/R}e^{ix/R-ift} \tag{7.7.21}$$

From (7.7.2), we get

$$u = \frac{gA}{C}e^{-y/R}e^{ix/R-ift} \tag{7.7.22}$$

This is called the inertial oscillation, which is a special case of Kelvin wave.

7.7.4 Kelvin Wave, $\sigma = \pm C_0 k$

$$\frac{\sigma}{k} = \text{phase velocity} = \pm C_0 = \pm\sqrt{gH_o}$$

This relation is the same as that for surface gravity waves. Let us focus attention to the rightward waves and take the plus sign. From Eqn. (7.7.14):

$$\alpha^2 = -\frac{f^2}{C_0^2} = -\frac{1}{R^2}$$

where

$$R = \frac{C_0}{f} \tag{7.7.23}$$

is defined to be the Rossby radius of deformation. Thus

$$\alpha = \pm\frac{i}{R}, \quad \frac{\sigma}{f} = \frac{kC_0}{f} = kR, \tag{7.7.24}$$

and the solution can be written as

$$\bar{\eta} = Ae^{-y/R} + Be^{y/R} \tag{7.7.25}$$

Now $Ae^{-y/R}$ satisfy the boundary condition (7.7.12)automatically for all y:

$$\left(\frac{d}{dy}+\frac{fk}{\sigma}\right)e^{-y/R}=\left(-\frac{f}{C_0}+\frac{f}{C_0}\right)e^{-y/R}=0$$

hence $B=0$, thus

$$\eta = Ae^{-y/R}e^{i(kx-\sigma t)} = Ae^{-fy/C_0}e^{i(kx-\sigma t)}. \qquad (7.7.26)$$

This is the Kelvin Wave. Water is piled up along the shore $y=0$ to the right of the wave vector. See Figure 7.7.1. A field record of English channel is in Figure 7.7.2.

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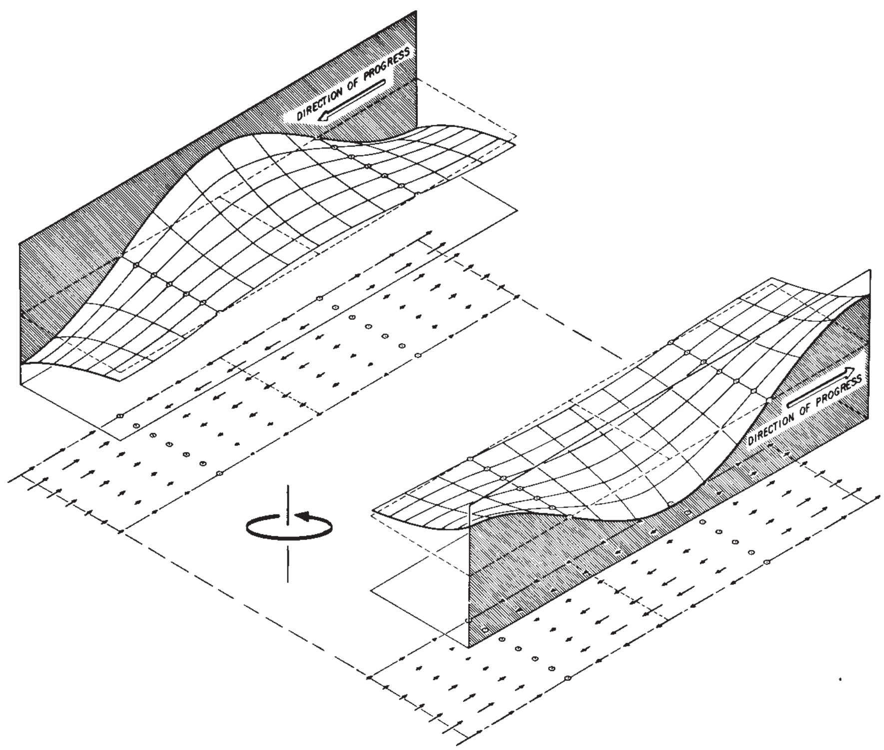

Fig. 10.3. Northern hemisphere Kelvin waves on opposite sides of a channel that is wide compared with the Rossby radius. In each vertical plane parallel to the coast, the currents (shown by arrows) are entirely within the plane and are exactly the same as those for a long gravity wave in a nonrotating channel. However, the surface elevation varies exponentially with distance from the coast in order to give a géostrophic balance. This means Kelvin waves move with the coast on their right in the northern hemisphere and on their left in the southern hemisphere. [From Mortimer (1977).]

Figure 7.7.1: Kelvin wave along a channel. From Gill

There are some more pecularities. From the momentum equations we have

$$\frac{\partial^2 v}{\partial t^2}+f^2 v = -g\left(\frac{\partial^2\eta}{\partial y\partial t}-f\frac{\partial\eta}{\partial x}\right)$$

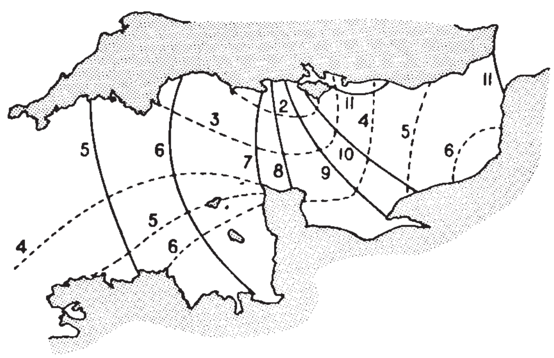

Fig. 10.5. Co-tidal lines (solid) with time in lunar hours, and co-range lines (dotted with values in meters) for the English Channel. [From Proudman (1953, p. 262); after Doodson and Corkan (1931).]

Figure 7.7.2: Kelvin wave in British Channel.

$$= \; -g\left(Ae^{ikx-i\sigma t}e^{-fy/C_0}\right)\left[\frac{-f}{C_0}(-i\sigma_0) - fik\right]$$
$$= \; -gAe^{ikx-\sigma t}e^{-fy/C_0}i\frac{f\sigma}{C_0}(1-1) = 0.$$

Therefore,

$$v = 0 \tag{7.7.27}$$

identically. Now the x-momentum equation and mass conservation equation reduce to

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{7.7.28}$$

and

$$\frac{\partial \eta}{\partial t} = -H_0\frac{\partial u}{\partial x} \tag{7.7.29}$$

These are formally the long wave equation in one space dimension x. But η and u depend on y!! Indeed for the propagating wave

$$\frac{\partial u}{\partial t} = -i\sigma u = -g\frac{\partial \eta}{\partial x} = -ikg\eta \qquad \sigma = C_0 k$$

hence

$$u = \frac{kg}{\sigma}\eta$$

From the solution (7.7.26),

$$\frac{\partial \eta}{\partial y} = \frac{-f}{C_0}\eta = \frac{-f}{C_0}\frac{\sigma u}{gk} = -\frac{fC_0}{g}.$$

Therefore,

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y}$$

This is a state of Quasi-static Geostrophy!

Note

$$R = \text{Rossby radius of deformation} = \frac{C_0}{f} = \left[\frac{\text{vel}}{1/\text{Time}}\right] = [\text{Length}]$$

If $H_0 = 30m$

$$C_0 = \sqrt{gH_0} = 1.732 \times 10m/s, f = 10^{-4}s^{-1}$$

$$R = \frac{C_0}{f} = 1.732 \times 10^5 m = 173km.$$

If $H_0 = 1000m$

$$R = 1000km.$$

7.7.5 Poincare waves

Consider the eigenvalue condition,

$$\sin \alpha L = 0 \qquad \alpha = \alpha_n = \frac{n\pi}{L} \qquad n = 0, 1, 2, 3, \cdots \tag{7.7.30}$$

From Eqn. (4.3)

$$\frac{\sigma_n^2 - f^2}{C_0^2} - k^2 = \left(\frac{n\pi}{L}\right)^2$$

or

$$\sigma_n = \pm\left\{f^2 + C_0^2\left[k^2 + \left(\frac{n\pi}{L}\right)^2\right]\right\}^{1/2}.$$

This relation between frequency and wave number $\sigma = \sigma(k)$ is called the dispersion relation.

The dispersion relation can also be written

$$\frac{\sigma_n}{f} = \pm\sqrt{1 + \left(\frac{C_0}{f}\right)^2\left[k^2 + \left(\frac{n\pi}{L}\right)^2\right]} = \pm\sqrt{1 + \left[(kR)^2 + \left(\frac{n\pi R}{L}\right)^2\right]}. \tag{7.7.31}$$

See Figure 7.7.3

The free surface of the $n-$th Poincare mode is:

$$\begin{aligned}\overline{\eta_n} &= B\left(\frac{A}{B}\sin\alpha_n y + \cos\alpha_n y\right)\\ &= B\left(-\frac{fk}{\sigma\alpha_n}\sin\alpha_n y + \cos\alpha_n y\right)\\ &= B\left(\cos\frac{n\pi y}{L} - \frac{L}{n\pi}\frac{fk}{\sigma_n}\sin\frac{n\pi y}{L}\right).\end{aligned}$$

or,

$$\eta_n = \eta_0\left(\cos\frac{n\pi y}{L} - \frac{L}{n\pi}\frac{fk}{\sigma_n}\sin\frac{n\pi y}{L}\right)\cos\left(kx - \sigma_n t\right) \qquad (7.7.32)$$

The veloctiy components are given by

$$u = \frac{\eta_0}{H_0}\left(\frac{C_0^2 k}{\sigma_n}\cos\frac{n\pi y}{L} - \frac{fL}{n\pi}\sin\frac{n\pi y}{L}\right)\cos(kx - \sigma_n t) \qquad (7.7.33)$$

$$v = -\frac{\eta_0}{H_0}\frac{L}{\sigma_n n\pi}\left(f^2 + \frac{C_0^2 n^2\pi^2}{L^2}\right)\sin\frac{n\pi y}{L}\ \sin(kx - \sigma_n t) \qquad (7.7.34)$$

Additional types of waves exist if the depth is not constant, or the water is stratified.

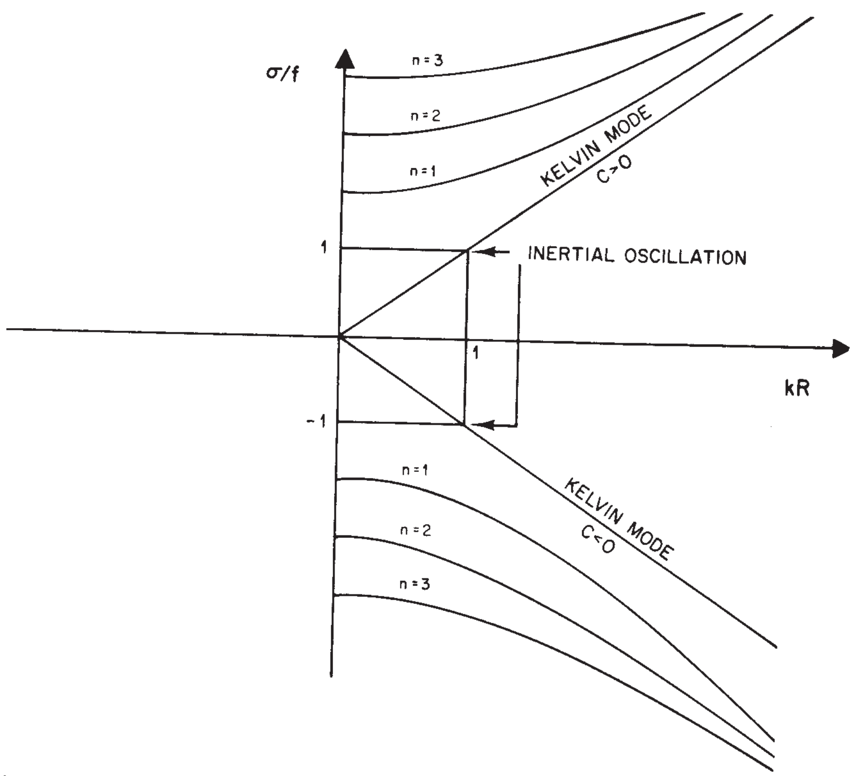

Figure 3.9.2 The dispersion diagram for Poincaré and Kelvin waves, showing the coincidence of the inertial oscillation $\sigma/f = \pm 1$ and the Kelvin mode at $kR = 1$.

Figure 7.7.3: Dispersion relation between frequency and wave nujmber.