Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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7.6 Transient longshore wind

[Ref]: Chapter 14, p. 195 ff, Cushman-Roisin
Csanady: Circulation in the Coastal Ocean

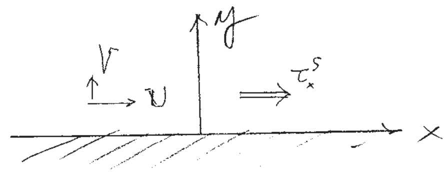

Figure 7.6.1: Longshore wind

Let the x be the coast, and the sea is in $y > 0$, as seen in Figure 7.6.1.

In view of the last section, we ignore the bottom stress. Assume that the wind is uniform in space but transient in time, so that $\partial/\partial x = 0$, The flux equations are

$$\frac{\partial \eta}{\partial t} + \frac{\partial V}{\partial y} = 0 \tag{7.6.1}$$

$$\frac{\partial U}{\partial t} - fV = \frac{\tau_x^S}{\rho} \tag{7.6.2}$$

$$\frac{\partial V}{\partial t} + fU = -gh\frac{\partial \eta}{\partial y}. \tag{7.6.3}$$

The boundary condition on the coast $x = 0$: $\quad U = 0$.

7.6.1 Sudden long-shore wind

Let the wind stress be

$$\tau_x^S = \begin{cases} 0, & t \le 0, \\ T, & t > 0. \end{cases} \tag{7.6.4}$$

the initial conditions are

$$\eta, U, V = 0, \quad t = 0, \quad \forall y. \tag{7.6.5}$$

This initial-boundary value problem can be solved by Laplace transform (Crépon, 1967). The solution consists of two parts: one part is oscillatory and decays with time; the other part increases monotonically with time. To avoid the complex mathematics we only examine the latter which is the dominant part for large time,

$$U = t\bar{U}(y), \quad V = \bar{V}(y), \quad \eta = t\bar{\eta}(y) \tag{7.6.6}$$

The oscillatory part is needed to ensure the initial condition on U. This solution clears fails for very large t, indicating the there is no steady- state solution in the linearized approximation. For a realistic theory nonlinearity must be accounted for, or the wind duration has to be finite.

It is easy to see from (7.6.1) to (7.6.3) that

$$\bar{\eta} + \frac{d\bar{V}}{dy} = 0 \tag{7.6.7}$$

$$f\bar{U} = -gh\frac{d\bar{\eta}}{dy} \tag{7.6.8}$$

$$\bar{U} - f\bar{V} = T/\rho \tag{7.6.9}$$

These three equations can be combined into one :

$$\frac{d^2\bar{V}}{dy^2} - \frac{f^2}{gh}\bar{V} = \frac{fT}{\rho gh} \tag{7.6.10}$$

The solution satisfies no flux on the coast is

$$\bar{V} = -\frac{T}{\rho f}\left(1 - e^{-x/R_o}\right) \tag{7.6.11}$$

where

$$R_o = \frac{\sqrt{gh}}{f} \tag{7.6.12}$$

is called the **Rossby radius of deformation**. If we take $f = 10^{-4}$ 1/s in a shallow sea of $h = 10$ m the Rossby radius is about 10^5 m $= 100$ km.

It is easy to find that

$$\eta = t\bar{\eta} = t\frac{T}{\rho gh}e^{-y/R_o} \tag{7.6.13}$$

and

$$U = t\bar{U} = t\frac{T}{\rho f}e^{-y/R_o} \tag{7.6.14}$$

Clearly when $y/R_o \gg 1$, the coast line has no influence. The flux is $U = 0, V = -T/\rho f$, and is inclined to the right of the wind by 90 degrees, as predicted by the Ekman layer theory. The sea surface rises if $T > 0$ (coast is on the right of wind), and sinks near the coast if $T < 0$ (coast is on the left of wind).

7.6.2 Sinusoidal wind stress

We now consider

$$\tau_x^S = \Re\left(i\tau_0 e^{-i\omega t}\right) = \tau_o \sin\omega t \tag{7.6.15}$$

Let

$$(\eta, U, V) = \Re\left[(\eta_0, U_0, V_0)\, e^{-i\omega t}\right] \tag{7.6.16}$$

The symbol $\Re$ (real part of) will be omitted for brevity, but must be remembered before numerical calculation.

Let us calculate the total flux (The boundary layers can be studied later.),

$$-i\omega\eta_0 + \frac{dV_0}{dy} = 0 \tag{7.6.17}$$

$$-i\omega U_0 - fV_0 = i\frac{\tau_0}{\rho}, \tag{7.6.18}$$

$$-i\omega V_0 + fU_0 = -gh\,\frac{d\eta_0}{dy} \tag{7.6.19}$$

An equation for a single variable can be obtained. For example by solving Eqns. (7.6.18) and (7.6.19) for U_0 and V_0, we get

$$U_0 = \frac{\begin{vmatrix} i\tau_0/\rho & -f \\ -gh\frac{d\eta_o}{dy} & -i\omega \end{vmatrix}}{\begin{vmatrix} -i\omega & -f \\ f & -i\omega \end{vmatrix}} = -\frac{\tau_0\omega/\rho - ghf\frac{d\eta_0}{dy}}{-\omega^2 + f^2} \tag{7.6.20}$$

and

$$V_0 = \frac{\begin{vmatrix} -i\omega & i\tau_0/\rho \\ f & -gh\frac{d\eta_o}{dy} \end{vmatrix}}{\begin{vmatrix} -i\omega & -f \\ f & -i\omega \end{vmatrix}} = \frac{-i\tau_0 f/\rho + i\omega gh\frac{d\eta_0}{dy}}{-\omega^2 + f^2} \tag{7.6.21}$$

Differentiate Eqn. (7.6.20) and use Eqn. (7.6.17)

$$-i\omega\eta_0 + \frac{i\omega gh}{f^2 - \omega^2}\frac{d^2\eta_0}{dy^2} = 0$$

or

$$\frac{d^2\eta_0}{dy^2} - \frac{f^2 - \omega^2}{gh}\eta_0 = 0 \tag{7.6.22}$$

We now distinguish two cases.

Low frequency: $\omega < f$

The solution to (7.6.22) bounded at infinity is

$$\eta_0 = A\, e^{-y/R_0} \tag{7.6.23}$$

where

$$R_0 = \sqrt{\frac{gh}{f^2 - \omega^2}}. \tag{7.6.24}$$

is the modified Rossby radius.

Applying the B.C. on the coast : $V_0 = 0, \;\; y = 0$, we get from (7.6.20),

$$\frac{d\eta_0}{dy} = \frac{f}{\rho g H}\,\frac{\tau_0}{\omega} \overset{(7.6.23)}{=} \frac{-A}{R_0}.$$

and,

$$A = -\frac{\tau_0}{\omega}\,\frac{f}{\rho g h}\, R_0 \tag{7.6.25}$$

Hence

$$\eta_0 = -\frac{f\tau_0}{\rho\omega g h}\, R_0\, e^{-y/R_o}$$

and finally

$$\eta = -\frac{f\tau_0}{\rho\omega g h}\, R_0\, e^{-y/R_0}\, e^{-i\omega t} \tag{7.6.26}$$

Its real part is the solution.

Now from Eqn. (7.6.20), we get

$$U = \frac{\tau_o \omega}{\rho(f^2 - \omega^2)} \left(1 - \frac{f^2}{\omega^2} e^{-y/R_0}\right) e^{-i\omega t}. \tag{7.6.27}$$

From Eqn. (7.6.21)

$$V_0 = -\frac{i\tau_0 f}{\rho(f^2 - \omega^2)} \left(1 - e^{-y/R_0}\right) \tag{7.6.28}$$

$$\tag{7.6.29}$$

Hence,

$$V = -\frac{\tau_0 f}{\rho\,(f^2 - \omega^2)} \left(1 - \frac{f^2}{\omega^2}\, e^{-x/R_0}\right) \sin \omega t. \tag{7.6.30}$$

If $\tau_0 > 0$ (wind blows eastward, along the coast), the sea level near the coast sinks at the start .

High frequency : $\omega > f$

Of the two possible oscillatory solutions to (7.6.22), we must choose the one that represents outgoing waves at infinty (the radiation condition),

$$\eta_0 = Be^{ikx}, \tag{7.6.31}$$

where the wavenumber is the inverse of the modified Rossby radius of deformation,

$$k = \sqrt{\frac{\omega^2 - f^2}{gh}} \tag{7.6.32}$$

It is easy to show that from the no-flux confditon on the coast thata

$$B = \frac{-i\tau_0 f}{\rho g h \omega k} \tag{7.6.33}$$

hence, in complex form,

$$\eta = -\frac{i\tau_0 f}{\rho g h \omega k} e^{ikx - i\omega t} \tag{7.6.34}$$

or, in real form,

$$\eta = -\frac{\tau_0 f}{\rho g h \omega k} \sin(kx - \omega t), \tag{7.6.35}$$

The final formulas for U, V are left as an exercise.
Homework: For the oscillating-wind problem, find the flow in the inviscid interior and then the flow in the surface boundary layer.