Lecture Notes on Fluid Dynamics
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7.4 Steady and uniform onshore wind in a shallow Sea

We now examine a few simple examples. In this section we consider an ideal situation where the horizontal length scale is infinite, so that only the vertical variations are essential. In the next section we demonstrate the effects of horizontal nonuniformity. In these examples small Rossby number is assumed from the outset.

Let us consider the effects of steady onshore wind on an infinitely long coast. Momentun transfer from the wind must go through shear, hence viscosity must te present. To be confirmed later, we expect the viscous effects to be important in thin boundary layers near the sea surface and the bottom. Let us augment the inviscid shallow water equations with the most important turbulent shear stress τ_{xz}, τ_{yz}. Horizontal shear stresses $\tau_{xy}, \tau_{xx}, \tau_{yy}$ can be ignored if we are sufficiently far away from the coast line. Using the crudest approximation of constant eddy viscosity, ν, the governing equation in a shallow sea are

$$\frac{\partial \eta}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{7.4.1}$$

where

$$U = \int_{-h}^{0} u dz, \qquad V = \int_{-h}^{0} v dz \tag{7.4.2}$$

are the depth-integrated transport rates in x, y directions.

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \tag{7.4.3}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial z} + \nu \frac{\partial^2 v}{\partial z^2} \tag{7.4.4}$$

The boundary conditons on the sea surface are

$$\mu \frac{\partial u}{\partial z} = \tau_x^S, \quad \mu \frac{\partial v}{\partial z} = \tau_y^S \tag{7.4.5}$$

On the seabed, tubulence effects are complicated so that the friction is often modeled by an empirical law:

$$\tau_x^B = C_f u\sqrt{(u^2+v^2)}, \quad \tau_y^B = C_f v\sqrt{(u^2+v^2)}, \tag{7.4.6}$$

These make the problem nonlinear. We shall study a simpler model which is more appropriate for the laboratory,

$$u = v = w = 0, \quad z = -h \tag{7.4.7}$$

Integrating over depth and defining the horizontal transport rates,

$$U = \int_{-h}^{0} u dz, \qquad V = \int_{-h}^{0} v dz \tag{7.4.8}$$

Note that U, V are fluxes with the dismension of(veloctiy)×(depth). We then get

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &+ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \\
\frac{\partial U}{\partial t} &- fV = -gh\frac{\partial \eta}{\partial x} + \frac{\tau_x^S}{\rho} - \frac{\tau_x^B}{\rho} \\
\frac{\partial V}{\partial t} &+ fU = -gh\frac{\partial \eta}{\partial y} + \frac{\tau_y^S}{\rho} - \frac{\tau_y^B}{\rho}.
\end{aligned}$$

where

$$\tau_x^S = \rho\nu\left[\frac{\partial u}{\partial z}\right]_{z=0}, \quad \tau_y^S = \rho\nu\left[\frac{\partial v}{\partial z}\right]_{z=0} \tag{7.4.9}$$

$$\tau_x^B = \rho\nu\left[\frac{\partial u}{\partial z}\right]_{z=-h}, \quad \tau_y^B = \rho\nu\left[\frac{\partial v}{\partial z}\right]_{z=-h} \tag{7.4.10}$$

As an order estimate, the scale of the horizontal flux is

$$[U] \sim \frac{\tau^S}{\rho f} \sim \frac{u_*^2}{f}$$

where u_* is the friction velocity. A vertical boundary layer (of Ekman) can exist wherein Coriolis force $\rho f U$ balances the viscous stress

$$\rho f U \sim \tau \sim \rho\nu\frac{\partial u}{\partial z} \sim \rho\nu\frac{U}{h}\frac{1}{\delta}$$

Therefore, the Ekman layer thickness is

$$\delta = O\left(\sqrt{\frac{\nu}{f}}\right)$$

A typical value of eddy viscosity is $\nu = 1\ cm^2/s$ and $f = 10^{-4}\ 1/s$. Therefore the Ekman layer thicknss is $O(1)$ m.

Our strategy is to get the horizontal transport, then the details of the boundary layers.

7.4.1 Wind setup due to steady onshore wind

Consider an infinitely long coastline along the x axis. The sea is on the side $y > 0$. Assume τ_y^S to be a given constant. Consider the steady state $\partial/\partial t = 0$ and ignore τ^B first[1]. Beginning from the equations :

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0.$$

[1]This will be checked later

$$
\begin{aligned}
-fV &= -gh\frac{\partial \eta}{\partial x} \\
+fU &= -gh\frac{\partial \eta}{\partial y} + \frac{\tau_y^S}{\rho}
\end{aligned}
$$

with

$$V = 0, \quad y = 0 \tag{7.4.11}$$

Uniformity in x implies $\partial\eta/\partial x = 0$ hence $V \equiv 0$ everywhere. Let us assume that far at $x \sim -\infty$, the coast is blocked by a continent. We must have

$$U = \text{constant} = 0, \quad \text{for all} x \tag{7.4.12}$$

Thus the total fluxes in the horizontal directions are zero. The only dynamics effect of wind is to induce a sea-level change: **wind Set-up**.

$$gh\frac{\partial \eta}{\partial y} = \frac{\tau_y^S}{\rho}. \tag{7.4.13}$$

$$
\begin{aligned}
\eta &= \frac{\tau_y^S}{\rho g h}y + \text{ constant} \\
&= \frac{\rho u_*^2}{\rho g h}y = \frac{u_*^2}{gh}y.
\end{aligned}
$$

If $u_* = 1$ cm/sec then $\tau_y^S = 0.1$ Pa. Take $\rho = 10^3\ kg/m^3$, $g = 10\ m/sec^2$ and $h = 30$ m

$$\frac{u_*^2}{gh} = \frac{(10^{-2})^2}{10\cdot 30} = 3\times 10^{-7}.$$

Note that

$$1\,\text{atm} = 10^5 N/m^2, \quad 1N/m^2 = 1Pa = \frac{1}{670}psi.$$

For $g = 10m/s^2$ $\qquad h = 30m$ the set up is calculated as follows.

τ_y^S	$\frac{\partial\eta}{\partial y}$	Δy	$\Delta\eta$
0.1 Pa	3×10^{-7}	100 km	3 cm
3 Pa	10^{-5}	300 km	3 m

Although there is no mean flow (or flux), there is internal flow. We now look at the detailed distribution in z, by deviding the depth into three parts: the geostrophic interior, the surface Ekman layer, and the bottom Ekman layer.

7.4.2 Geostrophic core

Outside the boundary layers, we have

$$-fv_g = -g\frac{\partial \eta}{\partial x} = 0, \tag{7.4.14}$$

$$fu_g = -g\frac{\partial \eta}{\partial y} = -\frac{u_*^2}{h}, \quad \text{or} \quad u_g = -g\frac{\partial \eta}{\partial y} = -\frac{u_*^2}{fh}, \tag{7.4.15}$$

In this geostropic balance, $v_g = 0$. There is a longshore current $u_g \neq 0$ in the core, pointing to the left of the wind. This solution cannot satisfy the sea-surface boundary condition. Boundary corrections are needed.

7.4.3 Surface Ekman layer

Now viscosity is important, so that the total velocity is governed by

$$\begin{aligned} -fv &= -g\,\frac{\partial \eta}{\partial x} + \nu\,\frac{\partial^2 u}{\partial z^2} \\ fu &= -g\,\frac{\partial \eta}{\partial y} + \nu\,\frac{\partial^2 v}{\partial z^2}. \end{aligned}$$

For this example

$$\frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \eta}{\partial y} = \frac{\tau_y^S}{\rho g h} = \frac{\rho u_*^2}{\rho g h} = \frac{u_*^2}{gh}$$

hence

$$-fv = \nu\,\frac{\partial^2 u}{\partial z^2} \tag{7.4.16}$$

$$fu = -\frac{u_*^2}{h} + \nu\,\frac{\partial^2 v}{\partial z^2}. \tag{7.4.17}$$

As long as $\delta/h \ll 1$, bounday-layer approximation can be made. Let us make some estimates based on empirical data, cited from Csanady :

$$\delta = 0.1\,\frac{u_*}{f}, \quad \nu = \frac{u_*^2}{200f}, \quad Re_* = \frac{u_*\delta}{\nu} = 20$$

Pedlosky :

$$\nu = 1 \sim 10^3 cm^2/sec$$

$$\delta = \sqrt{\frac{1 \sim 10^3}{10^{-4}}} cm = 10^2 cm \sim 3 \times 10^3 cm.$$

Let the total velocity in the surface boundary layer be

$$u = u_g + u_E = -\frac{u^{*2}}{fh} + u_E, \quad v = v_g + v_E = v_E. \tag{7.4.18}$$

so that (u_E, v_E) are the boundary layer corections. Then for $z < 0$, we have

$$-fv_E = \nu \frac{\partial^2 u_E}{\partial z^2} \tag{7.4.19}$$

$$fu_E = \nu \frac{\partial^2 v_E}{\partial z^2} \tag{7.4.20}$$

The boundary conditions are

$$\nu \frac{\partial u_E}{\partial z} = 0, \qquad \nu \frac{\partial v_E}{\partial z} = u_*^2 \qquad \text{on } z = 0$$

$$u_E,\ v_E \to 0 \qquad z \to -\infty.$$

This is the Ekman boundary-layer problem. The solution is best obtained by introducing the complex velocity,

$$q_E = u_E + iv_E$$

then

$$iq_E = \nu \frac{\partial^2 q_E}{\partial z^2} \quad \text{or} \quad \frac{d^2 q_E}{dz^2} - \frac{if}{\nu} q_E = 0$$

Let the solution be of the form,

$$q_E \propto e^{Dz}$$

then

$$D^2 - \frac{if}{\nu} = 0$$

Since

$$(i)^{1/2} = \pm e^{i\pi/4} = \pm \frac{1+i}{\sqrt{2}}.$$

We get

$$q_E = A \exp\left(\frac{1+i}{\sqrt{2}} \frac{z}{\sqrt{\nu/f}}\right) = A\, e^{(1+i)z/\delta}. \tag{7.4.21}$$

Let

$$\delta = \sqrt{\frac{2\nu}{f}} \tag{7.4.22}$$

denote the Ekman boundary layer thickness. Apply the boundary condition on the sea surface,

$$\nu \left.\frac{\partial q_E}{\partial z}\right|_0 = i\, u_*^2$$

hence

$$A = \frac{i\, u_*^2\, \delta}{(1+i)\nu}.$$

The solution is

$$q_E = u_E + i\, v_E = \frac{i\delta\, u_*^2}{(1+i)\nu}\, e^{(1+i)z/\delta} = \frac{\delta\, {u^*}^2}{2\nu}(1+i)\, e^{z/\delta}\left(\cos\frac{z}{\delta} + i\,\sin\frac{z}{\delta}\right) \tag{7.4.23}$$

Separating real and imaginary parts, we get the velocity components,

$$u_E = \frac{\delta}{2}\frac{u_*^2}{\nu}\, e^{z/\delta}\left[\cos\frac{z}{\delta} - \sin\frac{z}{\delta}\right] \tag{7.4.24}$$

$$v_E = \frac{\delta}{2}\frac{u_*^2}{\nu}\, e^{z/\delta}\left[\cos\frac{z}{\delta} + \sin\frac{z}{\delta}\right]. \tag{7.4.25}$$

Let us show the velocity vector at various height in a hodograph where the vector begins at the orgin. The position of the tip is a function of z. The trajectory of the tips is the hodograph. The hodograph is the spiral shown in Figure 7.4.1,

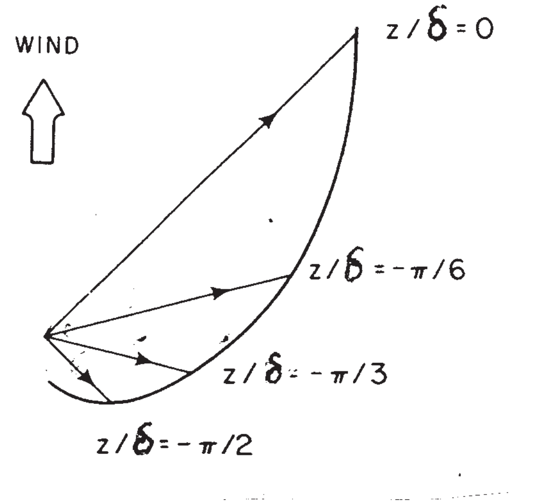

Figure 7.4.1: Hodograph in the Ekman boundary layer as a function of z/δ.

1. Physical Remark #1:
   Maximum velocity occurs on $z = 0$:

$$\frac{u_E(0)}{u_*} = \frac{v_E(0)}{u_*} = \frac{u_*}{f\delta}\left(\gg \frac{u_g}{u_*} = \frac{u_*}{fh}\right).$$

   and is 45 degrees to the right of wind.

2. Physical Remark #2 :
   The total flux in Ekman layer is

$$\begin{aligned} U_x^E + iV_y^E &= \int_{-\infty}^{0} dz\, (u_E + i\, v_E) = \int_{-\infty}^{0} dz\, q_E \\ &= \frac{\delta\, u_*^2}{2\nu}\,(1+i) \int_{-\infty}^{0} e^{(1+i)z/\delta} dz \\ &= \frac{\delta\, u_*^2}{2\nu}\,(1+i) \cdot \frac{\delta}{1+i} \cdot e^{(1+i)z/\delta}\Big|_{-\infty}^{0} = \frac{\delta^2\, u_*^2}{2\nu} = \frac{u_*^2}{f}. \end{aligned}$$

   where

$$\delta^2 = \frac{2\nu}{f}.$$

   Therefore the total mass flux in Ekman layer is 90 degrees inclined with respect to wind.

3. Physical remark # 3:
   Note that the flux in the surface Ekman layer is of the opposite sign as, hence is counter-balanced by, the geostrophic return flow beneath. Since u_g is very week, the velocity in the bottom Ekman layer must also be very weak. The flux thorugh the bottom boundary layer is therefore neglible, as will shall deduce below.

7.4.4 Bottom Ekman layer

The total flow is governed by

$$-fv = \nu \frac{\partial^2 u}{\partial z^2} \qquad fu = fu_g + \nu \frac{\partial^2 v}{\partial z^2}.$$

Let

$$u_E = u - u_g \qquad v_E = v \tag{7.4.26}$$

so that

$$-fv_E = \nu \frac{\partial^2 u_E}{\partial z^2}, \qquad fu_E = \nu \frac{\partial^2 v_E}{\partial z^2} \tag{7.4.27}$$

Let us shift to new coordinates with the origin on the sea bed so that the boundary conditions are

$$z \to \infty, \quad u_E, v_E \to 0 \tag{7.4.28}$$

and

$$z = 0, \quad u_E = -u_g, v_E = 0 \tag{7.4.29}$$

Let

$$q_E = u_E + iv_E \quad q_E = A\, e^{-(1+i)z/\delta} \tag{7.4.30}$$

Since there is no slip at $z = 0$

$$q_E = -u_g.$$

We conclude,

$$A = -u_g.$$

and

$$\begin{aligned} q_E &= (u_E + iv_E) = -u_g \, e^{-(1+i)z/\delta} \\ &= -u_G \, e^{-z/\delta} \left(\cos \frac{z}{\delta} - i \sin \frac{z}{\delta} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} u_E &= -u_g \, e^{-z/\delta} \cos \frac{z}{\delta} \\ v_E &= u_g \, e^{-z/\delta} \sin \frac{z}{\delta}. \end{aligned}$$

The bottom shear stress is

$$\nu \frac{\partial q_E}{\partial z} = \frac{1+i}{\delta} u_g \, e^{-(1+i)\, z/\delta}$$

at $z = 0$

$$\nu \left. \frac{\partial q_E}{\partial z} \right|_0 = \frac{1+i}{\delta} u_g = -\frac{1+i}{\delta} \frac{u_*^2}{fh}.$$

It is $O(1/\delta h f)$ times that of the surface stress. Using $u_* = 10^{-2}$ m/s, $\delta = 1$m, $h = 30$ m and $f = 10^{-4}$1/s, then $1/\delta f h \sim 1/30 \ll 1$. However if the wind stres is sufficiently strong, the bottom stress can be significant and the theory needs to be revised. Alternately the empirical boundary condition (7.4.6) should be used.

The total flux in bottom Ekman layer is

$$\begin{aligned} U_E + iV_E &= \int_0^\infty (u_E + iv_E) \, dz \\ &= -u_g \int_0^\infty e^{-(1+i/\delta)z} dz \\ &= -u_g \frac{1}{-(1+i)/\delta} \left. e^{-(1+i/\delta)z} \right|_0^\infty \\ &= -u_g \frac{\delta}{1+i} = \frac{u_*^2}{fh} \frac{\delta}{1+i} = \frac{\delta}{2}(1-i) \frac{u_*^2}{fh} \end{aligned}$$

It is of order δ and is directed at 135 degrees to the right of wind.

7.4.5 Summary

:

- Total Ekman layer flux on top is u_*^2/f
- Total geostropic flux is $-u_*^2/f$
- Total bottom Ekman layer flux is very small $O\left[\frac{u_*^2}{f}\left(\frac{\delta}{h}\right)\right]$.