Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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Chapter 7. Geophysical Fluid Dynamics of Coastal Region

[Ref]: Joan Brown and six others (Open Univeristy course team on oceanography): *Ocean Circulation*, Pergamon,1989.
Csanady: *Circulation in the Coastal Ocean*, Kluwer
Pedlosky *Geophysical Fluid Dynamics* Springer-Verlag
Murty : *Storm Surges*
Hill : *The Sea*, Chap 17, **Surges**, by P.Groen & G.W. Groves.

7.1 Introduction

Over a large spatical sclale (hundreds of kilometers or more), motion of sea water can be forced by wind, and affected by earth rotation. Wind owes its orgin to sun.
SUN, THE DRIVING FORCE:

1. induces temperature variation in the atmosphere, hence wind, which creates waves on the surface, and drives the ocean current.

2. Solar radiation, evaporation, precipitation, ice melting or freezing, etc cause temperature and salinity variation in the sea, hence stratification and thermohaline circulation.

EARTH ROTATION affects the ocean circulation:

Refering to Figure 7.1, let a missile be shot northward from a station on the equator of the rotating earth. While moving north, the missile initially has the same eastward velocity component as the earth, to a stationary observer in space. To an observer on the equator, however, the missle moves straight to the north. As the missile moves north, the ground moves beneath it. But the ground speed diminishes towards the north pole. To the earth beneath, the missile has an eastward velocity component, hence is deflected as if by an apparent force : Coriolis force. The apparent force increases as the missile goes further north.

Note:

- The Coriolis force acts at right angle to the direction of motion. If the missile is aimed to the north pole, i.e., traveling in the northern hemisphere, it will be deflected to the right of its path. If it is aimed to the south pole, the deflection is to the left.

- The Coriolis force increases from zero at the equator to the miximum at the north pole.

Fluid-mechanical problems of importance to coastal environment: Wind waves. Storm surges, Tides, waves and internal waves. Upwelling.

We now derive the Coriolis force mathematically.

7.2 Equations of Motion in Rotating Coordinates

Since the earth is rotating about the polar axis, the coordinate system fixed on earth is rotating. We need to know how to express the time rate of change of dynamical quantities in the rotating coordinates.

A vector fixed in the rotating coordinate system is rotating in the fixed (inertial) coordinate system. Consider therefore a vector rotating in the inertial frame of reference.

7.2.1 Vector of constant magnitude

If $\vec{A} = A_i\vec{e}_i$ has a constant magnitude but is rotating about an axis at the angular vecocity $\vec{\Omega}$, what is the rate of change $d\vec{A}/dt$ in the fixed coordinate (inertial) system? Let

$$d\vec{A} = \vec{A}(t+dt) - \vec{A}(t)$$

From Figure 7.2.2,

$$\left(\frac{d\vec{A}}{dt}\right)_I = \vec{e}|A| \sin\gamma \frac{d\theta}{dt},$$

where subscript I signifies "inertial system" and $\vec{e}$ is the unit-vector along $d\vec{A}$. Since $\vec{e} \perp \vec{A}$ and $\vec{e} \perp \vec{\Omega}$, we can write

$$\vec{e} = \frac{\vec{\Omega} \times \vec{A}}{|\,\vec{\Omega} \times \vec{A}\,|}.$$

and ,

$$\left(\frac{d\vec{A}}{dt}\right)_I = \frac{\vec{\Omega} \times \vec{A}}{|\,\vec{\Omega} \times \vec{A}\,|}\ |\,\vec{A}\,|\ \sin\gamma \frac{d\theta}{dt},$$

Since

$$\frac{d\theta}{dt} = \Omega,$$

$$|\,\vec{\Omega} \times \vec{A}\,| = \Omega\ |\,\vec{A}\,|\ \sin\gamma.$$

it follows that

$$\left(\frac{d\vec{A}}{dt}\right)_I = \vec{\Omega} \times \vec{A}. \tag{7.2.1}$$

In particular, let $\vec{A} = \vec{e}_i, i = 1, 2, 3$ be a base vector of unit length in the rotating frame of reference,

$$\vec{A} = \vec{e}_i$$

then

$$\left.\frac{d\vec{e}_i}{dt}\right|_I = \vec{\Omega} \times \vec{e}_i. \tag{7.2.2}$$

7.2.2 A vector of variable magnitude

Let

$$\vec{B} = B_i\,\vec{e}_i$$

be any non-constant vector in the rotating frame, and let

$$\left(\frac{d\vec{B}}{dt}\right)_R = \frac{dB_i}{dt}\,\vec{e}_i$$

denote its rate of change in the rotating frame. then

$$\left(\frac{d\vec{B}}{dt}\right)_I = \frac{dB_i}{dt}\,\vec{e}_i + B_i\,\frac{d\vec{e}_i}{dt} = \left(\frac{d\vec{B}}{dt}\right)_R + B_i\,\vec{\Omega}\times\vec{e}_i = \left(\frac{d\vec{B}}{dt}\right)_R + \vec{\Omega}\times\vec{B}. \tag{7.2.3}$$

Application (i): In particular, if $\vec{B} = \vec{r}$ is the position of a fluid particle, then the velociy of a fluid particle is

$$\left.\frac{d\vec{r}}{dt}\right|_I = \left.\frac{d\vec{r}}{dt}\right|_R + \vec{\Omega}\times\vec{r},$$

Note that $\vec{r}$ is the same in any coordinate system. Now $(d\vec{r}/dt)_I$ is the particle velocity seen in the inertial frame of reference and $(d\vec{r}/dt)_R$ is the particle velocity seen in the rotating frame of reference. Note that these time derivatives are Lagrangain (material) derivatives, hence

$$\vec{q} = \frac{d\vec{r}}{dt}$$

hence,

$$\vec{q}_I = \vec{q}_R + \vec{\Omega}\times\vec{r}; \tag{7.2.4}$$

Application (ii): Next we let $\vec{B} = \vec{q}_R$ be the velocity vector of fluid in the rotating frame of reference; its rates of change in the two frames of reference are related by

$$\left.\frac{d\vec{q}_R}{dt}\right|_I = \left.\frac{d\vec{q}_R}{dt}\right|_R + \vec{\Omega}\times\vec{q}_R. \tag{7.2.5}$$

Taking the Lagrangian time derivative of (7.2.4), and assuming that the angular acceleration of earth to be zero,

$$\frac{d\vec{\Omega}}{dt} = 0$$

we get

$$\begin{aligned}
\left(\frac{d\vec{q}_I}{dt}\right)_I &= \left(\frac{d\vec{q}_R}{dt}\right)_I + \vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_I \\
&= \left(\frac{d\vec{q}_R}{dt}\right)_R + \vec{\Omega}\times\vec{q}_R + \vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega}\times\vec{r}\right] \\
&= \left(\frac{d\vec{q}_R}{dt}\right)_R + \underbrace{2\vec{\Omega}\times\vec{q}_R}_{-\text{ Coriolis acc.}} + \underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{centripetal}}
\end{aligned} \tag{7.2.6}$$

The second term on the right is the negative of the Coriolis force, being perpendicular to both $\vec{q}$ and $\vec{\Omega}$. With the help of the left hand, one can see that the force is directed to the right of the velocity. The negative of the last term represents the centrifugal force

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = |\Omega|^2 \, \vec{r_\perp},$$

See Figure 7.2.3 for the geometric relations.

The centrifugal force may be written in terms of a centrifugal force potential ϕ_c where

$$\phi_c = \frac{1}{2} (\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r}) = \frac{1}{2} |\Omega|^2 r_\perp^2. \tag{7.2.7}$$

so that

$$\nabla \phi_c = \frac{d\phi_c}{dr_\perp} \vec{e}_\perp = |\Omega|^2 \, \vec{r_\perp}, \tag{7.2.8}$$

7.2.3 Summary of governing equations in rotating frame of reference:

Continuity:

$$\nabla \cdot \vec{q} = 0 \tag{7.2.9}$$

In the coordinate system rotating at the constant angular velocity, the momentum equation reads, after dropping subscripts R

$$\rho \left(\frac{d\vec{q}}{dt} + 2\vec{\Omega} \times \vec{q} \right) = -\nabla p + \rho \nabla (\phi_g + \phi_c) + \mu \nabla^2 \vec{q} \tag{7.2.10}$$

where

$$\phi_g = gz \qquad \phi_c = \frac{1}{2} \left(\vec{\Omega} \times \vec{r} \right) \cdot \left(\vec{\Omega} \times \vec{r} \right)$$

Remember that

$$\frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \nabla \vec{q}$$

is the Lagrangian derivative.

7.2.4 Dimensionless parameters

$$\frac{\frac{\partial \vec{q}}{\partial t}}{2\Omega \times \vec{q}} = \frac{O\left(\frac{U}{T}\right)}{\Omega U} = O\left(\frac{1}{\Omega T}\right)$$

$$\frac{\vec{q} \cdot \nabla \vec{q}}{2\vec{\Omega} \times \vec{q}} = \frac{U^2/L}{2\Omega U} = \frac{U}{2\Omega L} = \text{Rossby number}$$

$$
\begin{aligned}
\frac{\nu\nabla^2\vec{q}}{2\Omega\times\vec{q}} &= \frac{\nu U/L^2}{2\Omega U} = \frac{\nu}{2\Omega L^2} = \text{Ekman number} \\
\nabla\phi_g &= \vec{g} \\
\nabla\phi_c &= \Omega^2\vec{r}_L
\end{aligned}
$$

For numerical estimate, we take $\Omega = \frac{1}{12 \text{ hrs}} = 2.31\times10^{-5}s^{-1}$ and r = earth radius = 6400 km. Then $\omega^2 r \sim (2.31\times10^{-5})^2 \times 6.4\times10^6 \sim 3\times10^{-3}\text{m/s}^2$ while $g \sim 10\text{m/s}^2$. Hence $g \gg \Omega^2 r$; gravity is more important than centripetal force.

7.2.5 Coriolis accelertion in the shallow sea

We introduce the spherical polar coordinates as in the left of Figure 7.2.4, with θ being the latitude. In the northern hemisphere, $0<\theta<\pi/2$.

Refering to the right of Figure 7.2.4, the angular velocity of the earth is:

$$\vec{\Omega} = \vec{i}\,(-\Omega\cos\theta) + \vec{j}(0) + \vec{k}(\Omega\sin\theta)$$

The Coriolis force is

$$
2\vec{\Omega}\times\vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\Omega\cos\theta & o & 2\Omega\sin\theta \\ u & v & w \end{vmatrix}
$$

$$= \quad \vec{i}\,(-2\Omega v\sin\theta) + \vec{j}\,(2\Omega u\sin\theta + 2\Omega w\cos\theta) + \vec{k}\,(-2\Omega v\cos\theta)$$

Consider shallow water where the depth D is much less than the horizontal length L, i.e., $D \ll L$, and compare the two terms in the y direction of $(\vec{j})$

$$\frac{2\Omega w\cos\theta}{2\Omega u\sin\theta} = \frac{w}{u}\cot\theta = O\left(\frac{D}{L}\right)\cot\theta \ll 1$$

except along the equator where $\theta = 0$. We have used continuity so that $w/u = O(D/L)$.

In the vertical (z) direction along $(\vec{k})$,

$$\frac{-2\Omega v\cos\theta}{\frac{1}{\rho}\frac{\partial p_d}{\partial z}} = \frac{-2\Omega u\cos\theta}{\frac{L}{D}\frac{\partial p_d}{\partial x}} = \frac{D}{L}\frac{-2\Omega u\cos\theta}{\max\left(\frac{U}{T},\frac{U^2}{L},\Omega U\right)} = O\left(\frac{D}{L}\right) \ll 1$$

Hence in a shallow sea

$$2\vec{\Omega}\times\vec{q} \cong \vec{i}(-2\Omega v\sin\theta) + \vec{j}(2\Omega u\sin\theta)$$

Define

$$f = 2\Omega\sin\theta \tag{7.2.11}$$

to be the **Coriolis parameter** , then

$$2\vec{\Omega}\times\vec{q} \approx -fv\,\vec{i} + fu\,\vec{j} \tag{7.2.12}$$

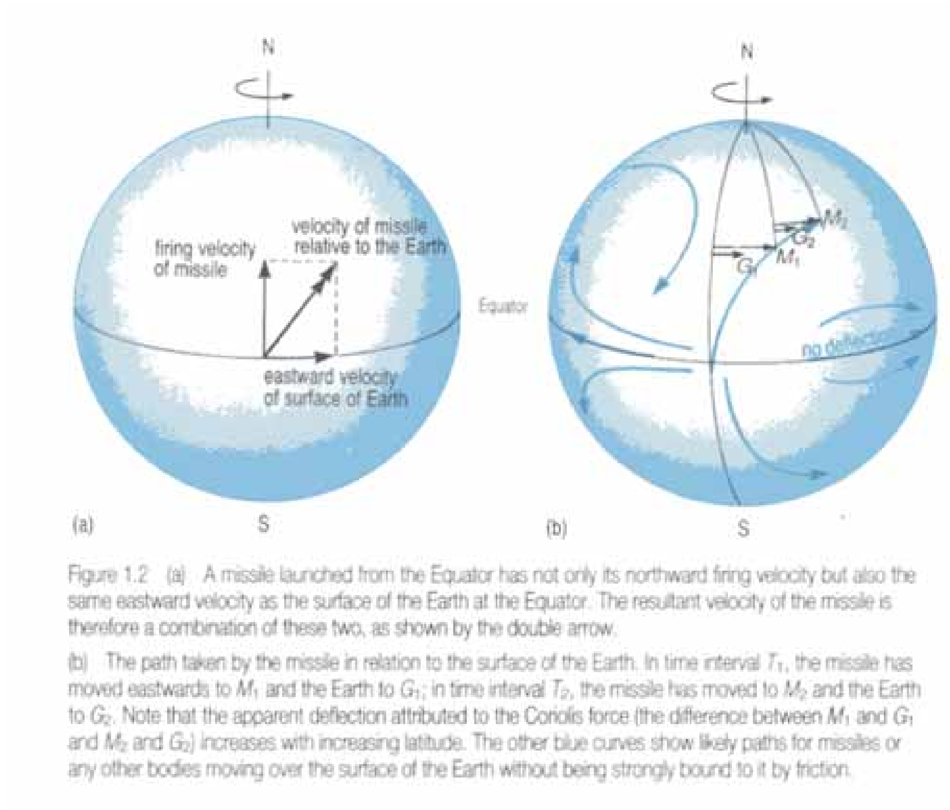

Figure 7.1.1: Effect of earth rotation. Copied from Brown et al, p 7.

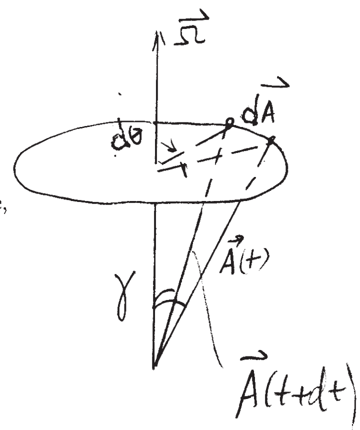

Figure 7.2.2: Vector $\vec{A}(t)$ rotating at the angular velocity $\vec{\Omega}$.

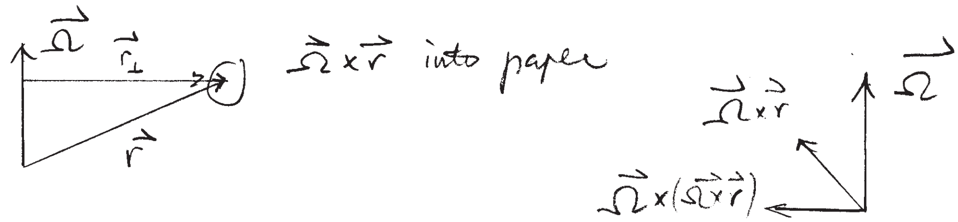

Figure 7.2.3: Coriolis force, position vector and angular velocity

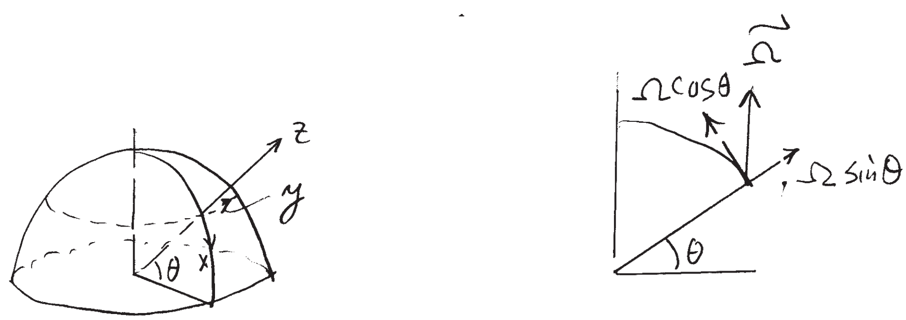

Figure 7.2.4: The Northern hemisphere.