Lecture notes in Fluid Dynamics
(1.63J/2.01J)
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4-6dispersion.tex March 12, 2007

[Refs]:

1. Aris:
2. Fung, Y. C. Biomechanics

4.7 Dispersion in an oscillatory shear flow

Relevant to the convective diffusion of salt and/or pollutants in a tidal channel, and chemicals in a blood vessel, Let us examine the Taylor dispersion in an oscillating flow in a pipe. Let the velocity profile be given,

$$u = U_s(r) + \Re\left[U_w(r)e^{-i\omega t}\right], \quad 0 < r < a. \tag{4.7.1}$$

The transport equation for the concentration of a solvent is recalled

$$\frac{\partial C}{\partial t} + \frac{\partial(uC)}{\partial x} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)\right) \tag{4.7.2}$$

Assume the pipe to be so small that diffusion affects the whole radius within one period or so, i.e.,

$$\tau_o \sim \frac{2\pi}{\omega} \sim \frac{a^2}{D} \tag{4.7.3}$$

We shall be interested in longitudinal diffusion across L much greater than a. Let U_o be the scale of U and

$$x = Lx',\ r = ar',\ u = U_o u',\ t = \frac{a^2}{D}t',\ \Omega = \frac{\omega a^2}{D} \tag{4.7.4}$$

Equation (4.7.2) is nomalized to

$$\frac{\partial C'}{\partial t'} + \frac{Ua}{D}\frac{a}{L}\frac{\partial(u'C')}{\partial x'} = \frac{a^2}{L^2}\frac{\partial^2 C'}{\partial x'^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) \tag{4.7.5}$$

Let the Péclét number be of order unity $Pe = Ua/D = O(a/L)^0$, (4.7.5) becomes

$$\frac{\partial C'}{\partial t'} + \epsilon Pe\frac{\partial(u'C')}{\partial x'} = \epsilon^2\frac{\partial^2 C'}{\partial x'^2} + \frac{1}{r}\frac{\partial}{\partial r'}\left(r'\frac{\partial C'}{\partial r'}\right) \tag{4.7.6}$$

with the boundary conditons

$$\frac{\partial C'}{\partial r'} = 0, \quad r' = 0, 1 \tag{4.7.7}$$

with

$$u' = U_s' + \Re U_w' e^{-i\Omega t'} \tag{4.7.8}$$

For brevity we drop the primes from now on.

4.7.1 Multiple scale analysis-homogenization

For convenience let us repeat the perturbation arguments of the last section.

There are three time scales : diffusion time across a, convection time across L, and diffusion time across L. Their ratios are :

$$\frac{a^2}{D} : \frac{L}{U_o} : \frac{L^2}{D} = 1 : \frac{1}{\epsilon} : \frac{1}{\epsilon^2}, \tag{4.7.9}$$

the smallest time scale being comparable to the oscillation period. Upon introducing the multiple time coordinates

$$t_0 = t,\ t_1 = \epsilon t,\ t_2 = \epsilon^2 t \tag{4.7.10}$$

and the multiple scale expansions.

$$C = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \ldots . \tag{4.7.11}$$

where $C_i = C_i(x, r, t_0, t_1, t_2)$, then the perturbation problems are

$O(\epsilon^0)$:

$$\frac{\partial C_0}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_0}{\partial r}\right) \tag{4.7.12}$$

with the boundary conditions:

$$\frac{\partial C_0}{\partial r} = 0, \quad r = 0, 1. \tag{4.7.13}$$

$O(\epsilon)$:

$$\frac{\partial C_0}{\partial t_1} + \frac{\partial C_1}{\partial t_0} + Pe\frac{\partial (uC_0)}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_1}{\partial r}\right) \tag{4.7.14}$$

with:

$$\frac{\partial C_1}{\partial r} = 0, \quad r = 0, 1. \tag{4.7.15}$$

$O(\epsilon^2)$:

$$\frac{\partial C_0}{\partial t_2} + \frac{\partial C_1}{\partial t_1} + \frac{\partial C_2}{\partial t_0} + Pe\frac{\partial (uC_1)}{\partial x} = \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_2}{\partial r}\right) \tag{4.7.16}$$

with

$$\frac{\partial C_2}{\partial r} = 0, \quad r = 0, 1. \tag{4.7.17}$$

Ignoring the transient that dies out quickly and focusing attention to the long-time evolution, i.e., $t_1 = O(1)$, the solution at $O(\epsilon^0)$ is [1]

$$C_0 = C_0(x, t_1, t_2), \tag{4.7.18}$$

[1]Strictly speaking the solution is

$$C_0 = C_{00}(x, t_0, t_1, t_2) + \sum_0^\infty C_{0n}(x, t_1, t_2) e^{-(k_n')^2 t_0} J_0(k_n' r)$$

where k_n' is the $n-$th root of $J_0'(ka) = 0$. The series terms die out quickly in $t_0 \gg 1$ and $t_1 \ll 1$, leaving the limit C_{00} which is independent of t_0. (Dr. E. Qian,1993)

At $O(\epsilon)$, let the known velocity be

$$u = U_s(y) + \Re\left(U_w(y)e^{-i\Omega t_0}\right) \tag{4.7.19}$$

then

$$\frac{\partial C_0}{\partial t_1} + \frac{\partial C_1}{\partial t_0} + Pe\left\{U_s + \Re\left[U(r)e^{-i\Omega t_0}\right]\right\}\frac{\partial C_0}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_1}{\partial r}\right) \tag{4.7.20}$$

Denoting the period average by overbars,

$$\bar{f} = \frac{\Omega}{2\pi}\int_{t_0}^{t_0+2\pi/\Omega} f\,dt_0$$

and taking the period average,

$$\frac{\partial C_0}{\partial t_1} + PeU_s\frac{\partial C_0}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{C}_1}{\partial r}\right) \tag{4.7.21}$$

with

$$\frac{\partial \bar{C}_1}{\partial r} = 0, \quad r = 0, 1 \tag{4.7.22}$$

Let us now integrate (or average) across the pipe, and get

$$\frac{\partial C_0}{\partial t_1} + Pe\langle U_s\rangle\frac{\partial C_0}{\partial x} = 0 \tag{4.7.23}$$

where angle brackets denote averaging over the cross section.

$$\langle h\rangle = \frac{1}{\pi}\int_0^1 2\pi r h\,dr$$

Now subtract (4.7.23) from (4.7.20)

$$\frac{\partial C_1}{\partial t_0} + Pe\left\{\tilde{U}_s + \Re\left[U_w e^{-i\Omega t_0}\right]\right\}\frac{\partial C_0}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_1}{\partial r}\right) \tag{4.7.24}$$

where

$$\tilde{U} = U_s(y) - \langle U_s\rangle \tag{4.7.25}$$

is the velocity nonuniformity

Now C_1 is governed by a linear equation, we can assume the solution to be proportional to the forcing and composed of a steady part and a time harmonic part, i.e.,

$$C_1 = Pe\frac{\partial C_0}{\partial x}\left\{B_s(r) + \Re\left[B_w(r)e^{-i\Omega t_0}\right]\right\} \tag{4.7.26}$$

then

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dB_s}{dr}\right) = \tilde{U}(r) \tag{4.7.27}$$

and

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dB_w}{dr}\right) + i\Omega B_w = U_w(r) \tag{4.7.28}$$

with the boundary conditions

$$\frac{dB_s}{dr} = 0 \quad \text{and} \quad \frac{dB_w}{dr} = 0,\ r = 0, 1. \tag{4.7.29}$$

After solving for B_s, B_w we go to $O(\epsilon^2)$, i.e., (4.7.16) :

$$\begin{aligned}\frac{\partial C_0}{\partial t_2} &+ \frac{\partial C_1}{\partial t_1} + \frac{\partial C_2}{\partial t_0} \\ &+ Pe^2\left\{\langle U_s\rangle + \tilde{U}_s + \Re\left[U_w e^{-i\Omega t_0}\right]\right\}\left\{B_s + \Re\left[B_w(y)e^{-i\Omega t_0}\right]\right\}\frac{\partial^2 C_0}{\partial x^2} \\ &= \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_2}{\partial r}\right)\end{aligned} \tag{4.7.30}$$

which is a linear PDE for C_2. From(4.7.26) and (4.7.23) we find

$$\frac{\partial C_1}{\partial t_1} = -Pe^2\frac{\partial^2 C_0}{\partial x^2}\langle U_s\rangle\left\{B_s(r) + \Re\left[B_w(r)e^{-i\Omega t_0}\right]\right\} \tag{4.7.31}$$

It follows that

$$\begin{aligned}\frac{\partial C_0}{\partial t_2} &+ \frac{\partial C_2}{\partial t_0} \\ &+ Pe^2\left\{\tilde{U}_s + \Re\left[U_w e^{-i\Omega t_0}\right]\right\}\left\{B_s + \Re\left[B_w(r)e^{-i\Omega t_0}\right]\right\}\frac{\partial^2 C_0}{\partial x^2} \\ &= \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_2}{\partial r}\right)\end{aligned} \tag{4.7.32}$$

Taking the time average over a period,

$$\frac{\partial C_0}{\partial t_2} + Pe^2\left\{\tilde{U}_s B_s + \frac{1}{2}\Re\left[U_w B_w^*\right]\right\}\frac{\partial^2 C_0}{\partial x^2} = \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{C}_2}{\partial r}\right) \tag{4.7.33}$$

with

$$\frac{\partial \bar{C}_2}{\partial r} = 0 \quad r = 0, 1 \tag{4.7.34}$$

Averaging (4.7.33) across the pipe, we get

$$\frac{\partial C_0}{\partial t_2} = E\frac{\partial^2 C_0}{\partial x^2} \tag{4.7.35}$$

with

$$E = 1 - Pe^2\left\{\langle \tilde{U}_s B_s\rangle + \frac{1}{2}\Re\langle U_w B_w^*\rangle\right\} \tag{4.7.36}$$

which is the effective diffusion coefficient or the dispersion coefficient. The first part is of molecular origin; the second part is due to fluid shear.

Finally we add (4.7.23) and (4.7.35) to get:

$$\left(\frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2}\right) C_0 + \epsilon Pe\langle U_s\rangle\frac{\partial C_0}{\partial x} = \epsilon^2 E\frac{\partial^2 C_0}{\partial x^2} \tag{4.7.37}$$

This describes the convective diffusion of the area averaged concentration, which is certainly of practical value.

After the perturbation analysis is complete, there is no need to use multiple scales; we may now write

$$\frac{\partial C_0}{\partial t_1} + Pe\langle U_s\rangle\frac{\partial C_0}{\partial x} = \epsilon E\frac{\partial^2 C_0}{\partial x^2} \tag{4.7.38}$$

in dimensionless form, or,

$$\frac{\partial C_0}{\partial t} + \langle U_s\rangle\frac{\partial C_0}{\partial x} = DE\frac{\partial^2 C_0}{\partial x^2} \tag{4.7.39}$$

in physical form. This equation governs the convective diffusion of the cross-sectional average, after the initial transient is smoothed out.

4.7.2 Aris' solution for a cicular pipe

R. Aris (1960, Proc Roy. Soc. Lond. 259, pp 370-376) has worked ouyt the solanwers for a flow forced by a periodic pressure gradient in a circular pipe of radius a. The analysis is carried out by using Bessel functions; only the solution is cited here.

For the pressure gradient

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = P\left[1 - \frac{1}{8}\left(a_n\cos\frac{2n\pi t}{T} + b_n\sin\frac{2n\pi t}{T}\right)\right] \tag{4.7.40}$$

First the velocity profile is found. Denoting

$$\langle U_s\rangle = \frac{Pa^2}{8\nu}, \quad Pe = \frac{\langle U_s\rangle a}{D} \tag{4.7.41}$$

$$\alpha_n = \frac{2n\pi a^2}{\nu T}, \quad \beta_n = \frac{2n\pi a^2}{DT}, \tag{4.7.42}$$

The dispersion corfficient is found to be

$$E = 1 + \frac{Pe^2}{48} + \frac{Pe^2}{8}\sum_1^\infty L(\alpha_n, \beta_n)(a_n^2 + b_n^2) \tag{4.7.43}$$

Denoting

$$\sigma = \frac{\alpha_n}{\beta_n} = \sqrt{\frac{D}{\nu}}, \quad y = \beta_n, \tag{4.7.44}$$

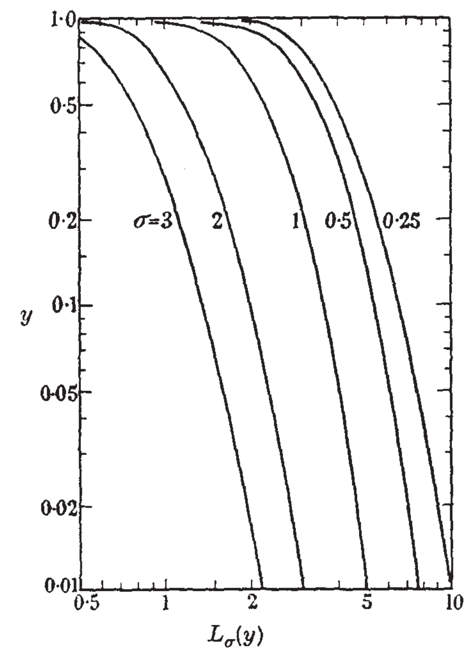

FIGURE 1. The coefficient function $L_\sigma(y)$.

Figure 4.7.1: The function $L_\sigma(y)$, Aris, 1960.

we can rewrite $L(x, y) = L(\sigma y, y) = L_\sigma(y)$ so that L is a function of y and the Schmitt number σ, as plotted in Figure 4.7.1.

Homework: Find the dispersion coefficient E in the oscillatory flow in a circular pipe and carry out the necesary numerical calculations.

Homework (mini research) : Find the dispersion coefficient E in the oscillatory flow in a blood vessel with elastic wall.