Lecture notes in Fluid Dynamics
(1.63J/2.01J)
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4.4 Buoyant plume from a steady heat source

[Reference]: Gebhart, et. al. (Jalluria, Maharjan, Saammakia), **Buoyancy-induced Flows and Transport**, 1988, Hemisphere Publishing Corporation

Let $\tilde{T} = T - T_\infty$ = temperature variation where T_∞ is a constant (no ambient stratification). For a strong enough heat source, we expect the boundary layer behavior,

$$\frac{\partial}{\partial r} \gg \frac{\partial}{\partial x}, \; u \gg v, \; \frac{\partial p}{\partial r} \cong 0$$

The boundary layer equations are

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0 \tag{4.4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = g\beta(T - T_\infty) + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{4.4.2}$$

$$u\frac{\partial \tilde{T}}{\partial x} + v\frac{\partial \tilde{T}}{\partial r} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}}{\partial r}\right) \tag{4.4.3}$$

The centerline $r = 0$ is an axis of symmetry,

$$v = \frac{\partial u}{\partial r} = \frac{\partial \tilde{T}}{\partial r} = 0 \tag{4.4.4}$$

Far outside the plume $r \to \infty$

$$u \to 0 \text{ and } T \to T_\infty, (\tilde{T} \to 0) \tag{4.4.5}$$

Rewrite (4.4.3) as

$$\frac{\partial(ru\tilde{T})}{\partial x} + \frac{\partial(rv\tilde{T})}{\partial r} - \tilde{T}\left(\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r}\right)$$
$$= \frac{\partial(ru\tilde{T})}{\partial x} + \frac{\partial(rv\tilde{T})}{\partial r} = k\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{T}}{\partial r}\right) \tag{4.4.6}$$

after using continuity. Now integrating the last equation from $r = 0$ to $r = \infty$

$$\frac{\partial}{\partial x}\int_0^\infty 2\pi r u\tilde{T} dr + 2\pi\int_0^\infty \frac{\partial(rv\tilde{T})}{\partial r} dr = k2\pi\int_0^\infty dr \frac{\partial}{\partial r}\left(r\frac{\partial\tilde{T}}{\partial r}\right)$$

therefore

$$\frac{\partial}{\partial x}\int_0^\infty 2\pi r u\tilde{T} dr + 2\pi r v\tilde{T}\Big|_0^\infty = 2\pi k\left(r\frac{\partial\tilde{T}}{\partial r}\right)_{r=0}^{r=\infty} \tag{4.4.7}$$

Using the boundary conditions, we get or

$$\int_0^\infty 2\pi r u\tilde{T} dr = \text{constant}$$

Note that

$$\begin{aligned}\int_0^\infty 2\pi r dr\, u\rho C\tilde{T} &= \text{rate of buoyancy flux}\\ &= \text{rate of heat flux}\\ &= Q(\text{given rate of heat release at } x = 0)\end{aligned}$$

therefore,

$$Q = \int_0^\infty 2\pi r dr \rho u C\tilde{T} \tag{4.4.8}$$

This is a boundary condition.

Let the stream function ψ be defined by

$$ru = \frac{\partial\psi}{\partial r}, \qquad rv = -\frac{\partial\psi}{\partial x} \tag{4.4.9}$$

(4.4.1) is automatically satisfied. From the momentum equation:

$$\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\frac{1}{r}\frac{\partial^2\psi}{\partial x\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = g\beta\tilde{T} + \frac{\nu}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right] \tag{4.4.10}$$

From the energy equation

$$\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\tilde{T}}{\partial x} - \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial\tilde{T}}{\partial r} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{T}}{\partial r}\right) \tag{4.4.11}$$

and from the buoyancy flux condition

$$Q = 2\pi\rho C\int_0^\infty r dr\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\tilde{T} \tag{4.4.12}$$

Try a similarity solution with the one-parameter transformation

$$x - \lambda^a x^*, \quad r = \lambda^b r^*, \quad \psi = \lambda^c \psi^*, \quad \tilde{T} = \lambda^d T^*$$

From (4.4.10),

$$\lambda^{2c-4b-a} = \lambda^{2c-4b-a} = \lambda^d = \lambda^{c-4b} \tag{4.4.13}$$

from (4.4.11)

$$\lambda^{c+d-2b-a} = \lambda^{d-2b} \tag{4.4.14}$$

and from (4.4.12)

$$\lambda^{c+d} = 1 \tag{4.4.15}$$

From these three equations we get

$$\frac{c}{a} = 1, \quad \frac{b}{a} = \frac{1}{2}, \quad \frac{d}{a} = -1.$$

We leave it as an exercise to show that the similarity variable can be taken to be

$$\eta = \frac{r}{x^{1/2}} \tag{4.4.16}$$

and the similarity solutions to be

$$\psi = xF(\eta), \quad \text{and} \quad \tilde{T} = x^{-1}G(\eta) \tag{4.4.17}$$

After much algebra, and noting

$$\frac{\partial \eta}{\partial r} = \frac{1}{x^{1/2}}, \quad \frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{r}{x^{3/2}} = -\frac{1}{2}\frac{r}{x^{1/2}}\frac{1}{x} = -\frac{\eta}{2x}$$

we get from (4.4.10)

$$\nu F''' + \left(\frac{F'}{\eta}\right)'(F - \nu) + g\beta\eta G = 0 \tag{4.4.18}$$

and from (4.4.11)

$$k(\eta G')' + (FG)' = 0 \tag{4.4.19}$$

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Before integrating, let us normalize :

$$\eta = \alpha\bar{\eta}, \quad F = \gamma\bar{F}, \quad G = \sigma\bar{G}. \tag{4.4.20}$$

It follows from (4.4.18) that

$$\frac{\nu\gamma}{\alpha^3}\bar{F}''' + \frac{\gamma}{\alpha^3}\left(\frac{\bar{F}'}{\bar{\eta}}\right)'(\gamma\bar{F} - \nu) + g\beta\alpha\sigma\bar{\eta}\bar{G} = 0 \tag{4.4.21}$$

where prime denotes $d/d\bar{\eta}$. Setting $\gamma = \nu$ and

$$\frac{\nu^2}{\alpha^3} = g\beta\alpha\sigma$$

which relates σ and α,

$$\sigma = \frac{\nu^2}{g\beta\alpha^4} \tag{4.4.22}$$

we get

$$\bar{F}''' + \left(\frac{\bar{F}'}{\bar{\eta}}\right)'(\bar{F} - 1) + \bar{\eta}\bar{G} = 0 \tag{4.4.23}$$

Similar normalization of (4.4.19) gives

$$\frac{k\alpha\sigma}{\alpha^2}(\bar{\eta}\bar{G}')' + \frac{\gamma\sigma}{\alpha}(\bar{F}\bar{G})' = 0 \tag{4.4.24}$$

. which can be simplified to

$$(\bar{\eta}\bar{G}')' + P_r(\bar{F}\bar{G})' = 0 \tag{4.4.25}$$

. where

$$P_r = \frac{\nu}{k} = \text{Prandtl Number} \tag{4.4.26}$$

For water $\nu = 10^{-2}cm^2/s, k = 1.42cm^2/s$, hence $Pr = 7$. For air $\nu = 0.145cm^2/s, k = 0.202cm^2/s$, hence $Pr = 0.75$.

We now integrate (4.4.25)to give

$$\bar{\eta}\bar{G}' + P_r\bar{F}\bar{G} = \text{constant}$$

Since $\psi(x, 0) = 0$, we must have $\bar{F}(0) = 0$; the constant above is zero.

$$\bar{\eta}\bar{G}' + P_r\bar{F}\bar{G} = 0 \tag{4.4.27}$$

Equation (4.4.27) can be written

$$\frac{\bar{G}'}{\bar{G}} = -P_r\frac{\bar{F}}{\bar{\eta}}, \quad \text{or} \quad \frac{d\ \ln\ \bar{G}}{d\bar{\eta}} = -P_r\frac{\bar{F}}{\bar{\eta}}$$

$$\ln\ \bar{G} = -P_r\int_0^{\bar{\eta}}\frac{\bar{F}}{\bar{\eta}}d\bar{\eta} + \text{constant}$$

$$\bar{G}(\bar{\eta}) = \bar{G}(0)\exp\left(-P_r\int_0^{\bar{\eta}}\frac{\bar{F}}{\bar{\eta}}d\bar{\eta}\right) \tag{4.4.28}$$

Substituting Eqn. (4.4.28) into Eqn. (4.4.23), the resulting equation for $\bar{F}$ must be integrated numerically.

Now let us find the boundary condtions for F or $\bar{F}$.

Eqn. (4.4.8) becomes

$$\frac{Q}{2\pi\rho C} = \int_0^\infty dr\, r \left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) \frac{G(\eta)}{x} = \int_0^\infty dr \frac{r}{r} x^{1/2} F' \frac{G}{x} = \int_0^\infty d\eta (F'G) = \nu\sigma \int_0^\infty d\bar{\eta} (\bar{F}'\bar{G}) \tag{4.4.29}$$

Therefore,

$$\int_0^\infty d\bar{\eta}\, \bar{F}'\bar{G} = \frac{Q}{2\pi\rho C\nu\sigma} \tag{4.4.30}$$

Let us choose

$$\frac{Q}{2\pi\rho C\nu\sigma} = 1 \tag{4.4.31}$$

so that

$$\int_0^\infty d\bar{\eta}\, \bar{F}'\bar{G} = 1 \tag{4.4.32}$$

is the boundary condition for $\bar{F}$ and $\bar{G}$. Now (4.4.31) defines σ, the scale of G. Note that larger Q implies larger σ and smaller α. Thus a stronger heat source leads to a greater centerline temperature and a thinner plume. Also,

$$u \to 0 \quad \text{as} \quad r \to \infty$$

hence

$$u = \frac{1}{r}\psi_r = \frac{F'}{\eta} = \frac{\nu}{\alpha^2}\frac{\bar{F}'}{\bar{\eta}} \to 0, \quad \text{as} \quad \eta \sim \bar{\eta} \to \infty$$

The radial velocity is, in general

$$v = \frac{1}{r}\psi_x = \frac{1}{r}\left(F - \eta\frac{F'}{2}\right)$$

Since

$$v \to 0 \qquad \text{as} \quad \eta \to 0,$$

we must have,

$$F(0) = 0.$$

Clearly

$$\bar{F}(\bar{\eta}) = 0 \quad \text{as} \quad \bar{\eta} \to 0 \tag{4.4.33}$$

The numerical results by Mollendorf & Gelhart, 1974, are shown in Figs. 4.4.1, for various Prandtl numbers. A schlierian photograph due to Gebhart (copied from Van Dyke **An Album of Fluid Motion**) is hown in Figure fig:plumeVD.

Remark:

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{F'}{\eta}\left(= \frac{x}{r}F'\frac{1}{x^{1/2}}\right)$$

Along the centerline $u(x,0) = \left(\frac{F'}{\eta}\right)_0$ = constant depending on P_r. Why? Buoyancy acceleration is counteracted by entrainment.

Remark: Let the radius of the plume be a which varies as

$$a \sim x^{1/2}$$

This is consistent with the behavior that $u \sim x^0$, and $\tilde{T} \sim x^{-1}$, since

$$a^2 u \tilde{T} = Q$$

On the other hand the mass flux rate is

$$u a^2 \sim x$$

and the momentum flux rate is

$$u^2 a^2 \sim x$$

hence both approach zero at the source. Thus a plume is the result of energy source, not of mass or momentum.

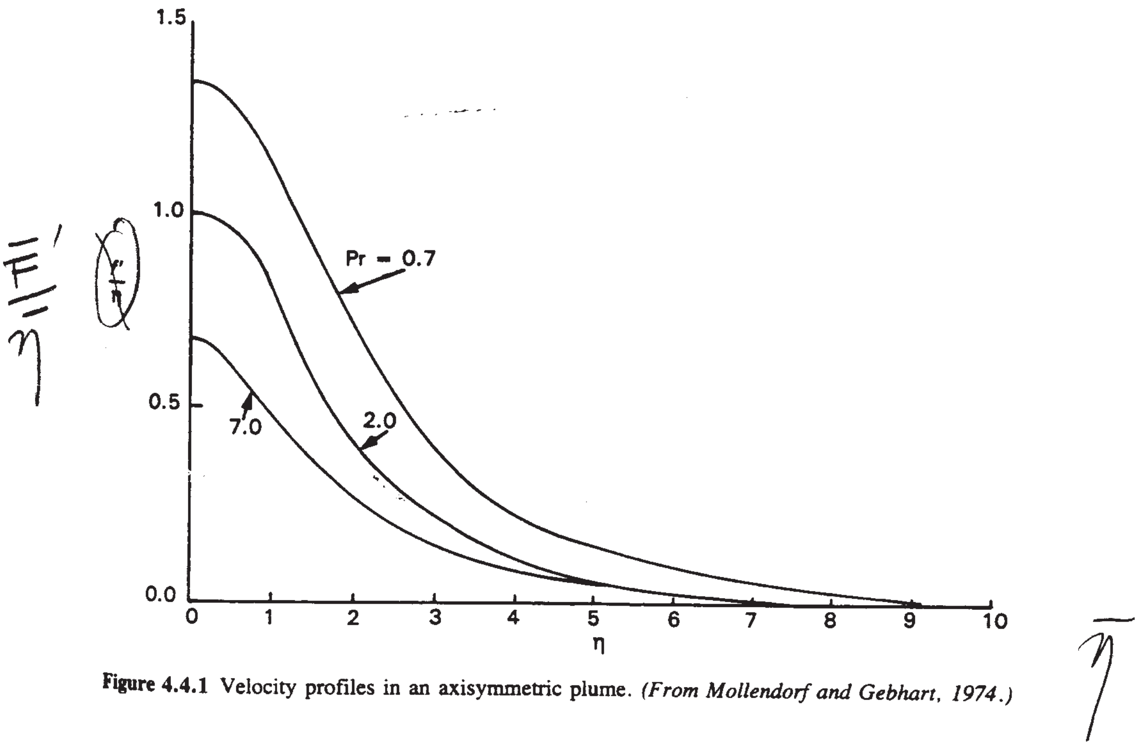

Figure 4.4.1 Velocity profiles in an axisymmetric plume. *(From Mollendorf and Gebhart, 1974.)*

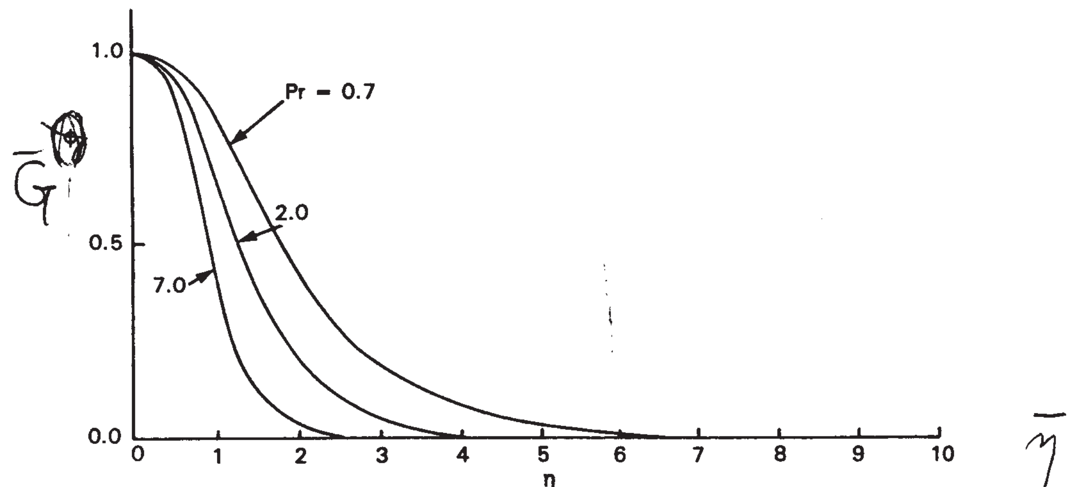

Figure 4.4.2 Temperature profiles in an axisymmetric plume. *(From Mollendorf and Gebhart, 1974.)*

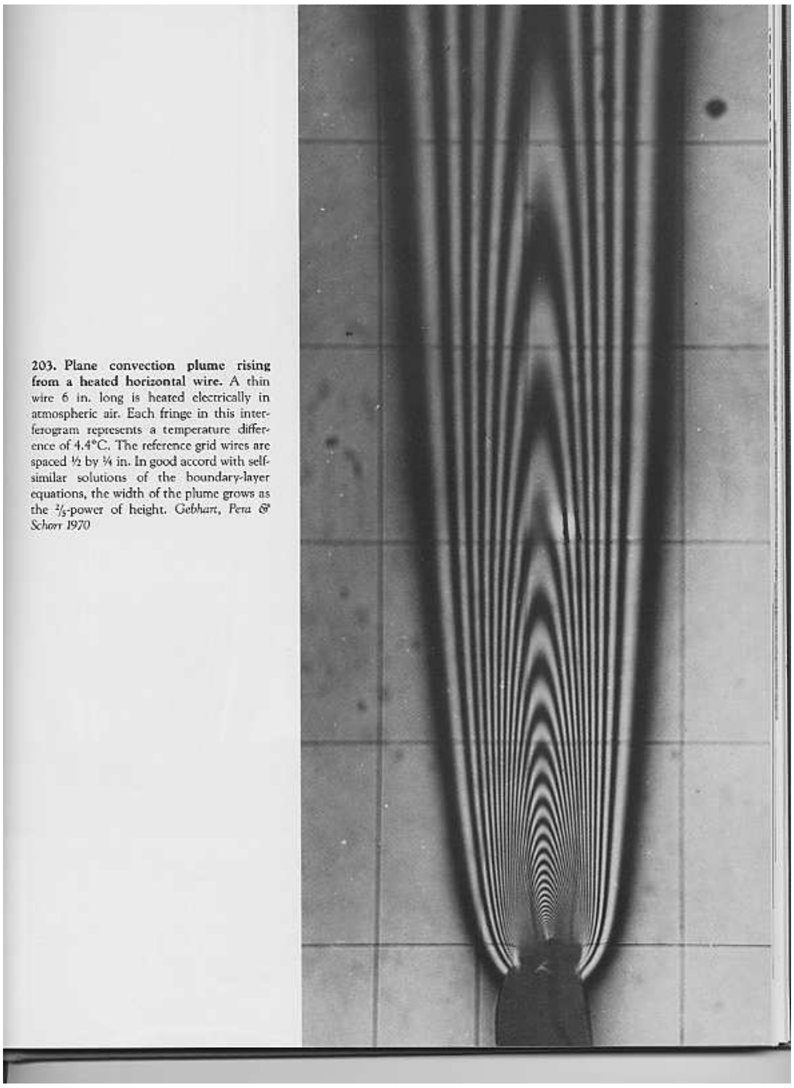

Figure 4.4.2: A 2D thermal plume from a line heat source. From Van Dyke, photo by Gebhart, Pera and Schoor 1970,