Lecture notes in Fluid Dynamics
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CHAPTER 4. THERMAL EFFECTS IN FLUIDS

4-1-2energy.tex

4.1 Heat and energy conservation

Recall the basic equations for a compressible fluid. Mass conservation requires that :

$$\rho_t + \nabla \cdot \rho \vec{q} = 0 \qquad (4.1.1)$$

Momentum conservation requires that :

$$\rho \left(\vec{q_t} + \vec{q} \nabla \cdot \vec{q} \right) = -\nabla p + \nabla \cdot \bar{\bar{\tau}} + \rho \vec{f} \qquad (4.1.2)$$

where the viscous stress tensor $\bar{\bar{\tau}}$ has the components

$$\left(\bar{\bar{\tau}} \right)_{ij} = \tau_{ij} = \mu \left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} \right) + \lambda \frac{\partial q_k}{\partial x_k} \delta_{ij}$$

There are 5 unknowns ρ, p, q_i but only 4 equations. One more equation is needed.

4.1.1 Conservation of total energy

Consider both mechanical ad thermal energy. Let e be the internal (thermal) energy per unit mass due to microscopic motion, and $q^2/2$ be the kinetic energy per unit mass due to macroscopic motion. Conservation of energy requires

$$\frac{D}{Dt} \iiint_V \rho \left(e + \frac{q^2}{2} \right) dV \qquad \text{rate of incr. of energy in } V(t)$$

$$= - \iint_S \vec{Q} \cdot \vec{n} \, dS \qquad \text{rate of heat flux into } V$$

$$+ \iiint_V \rho \vec{f} \cdot \vec{q} \, dV \qquad \text{rate of work by body force}$$

$$+ \iint_X \vec{\Sigma} \cdot \vec{q} \, dS \qquad \text{rate of work by surface force}$$

Use the kinematic transport theorm, the left hand side becomes

$$\iiint_V \rho \frac{D}{Dt} \left(e + \frac{q^2}{2} \right) dV$$

Using Gauss theorem the heat flux term becomes

$$-\iint_S Q_i n_i dS = -\iiint_V \frac{\partial Q_i}{\partial x_i} dV$$

The work done by surface stress becomes

$$\iint_S \Sigma_j q_j \, dS = \iint_S (\sigma_{ji} n_i) q_j \, dS$$

$$= \iint_S (\sigma_{ij} q_j) n_i \, dS = \iiint_V \frac{\partial(\sigma_{ij} q_j)}{\partial x_i} \, dV$$

Now all terms are expressed as volume integrals over an **arbitrary** material volume, the following must be true at every point in space,

$$\rho \frac{D}{Dt}\left(e + \frac{q^2}{2}\right) = -\frac{\partial Q_i}{\partial x_i} + \rho f_i q_i + \frac{\partial(\sigma_{ij} q_i)}{\partial x_j} \tag{4.1.3}$$

As an alternative form, we differentiate the kinetic energy and get

$$\rho \frac{De}{Dt} + \underbrace{\rho q_i \frac{Dq_i}{Dt}} = \underbrace{\rho f_i q_i + q_i \frac{\partial \sigma_{ij}}{\partial x_j}} - \frac{\partial Q_i}{\partial x_i} - p\frac{\partial q_i}{\partial x_i} + \tau_{ij}\frac{\partial q_i}{\partial x_j} \tag{4.1.4}$$

Becasue of momentum conservation, the terms in the underbraces cancel, leaving

$$\boxed{\rho \frac{De}{Dt} = -\frac{\partial Q_i}{\partial x_i} - p\frac{\partial q_i}{\partial x_i} + \tau_{ij}\frac{\partial q_i}{\partial x_j}} \tag{4.1.5}$$

We must now add Fick's law of heat conduction

$$Q_i = -K\frac{\partial T}{\partial x_i} \tag{4.1.6}$$

where K is the heat conductivity, and the following equations of state

$$e = e(p, T) \tag{4.1.7}$$

$$\rho = \rho(p, T) \tag{4.1.8}$$

Now there are 10 unknowns q_i, Q_i, ρ, p, e, T, and 10 equations: 1 from (4.1.1), 3 from (4.1.2), 1 from (4.1.5), 3 from (4.1.6), 1 from (4.1.7) and 1 from (4.1.8).

4.1.2 Equations of state:

For a perfect gas:

$$e = C_v T \tag{4.1.9}$$

$$p = \rho R T, \quad \text{where} \quad R = C_p - C_v. \tag{4.1.10}$$

The specific heats C_v (constant volume) and C_p (constant pressure) are measured in Joules /kg -dyne.

For a liquid:

$$e = CT \tag{4.1.11}$$

$$\rho = \bar{\rho}_o\left(\bar{T}_o, \bar{p}_o\right) + \left.\frac{\partial \rho}{\partial p}\right|_T \Delta p + \left.\frac{\partial \rho}{\partial T}\right|_p \Delta T + \cdots$$

where $\bar{\rho}_o, \bar{T}_o, \bar{p}_o$ are some constant reference density, temperature, and pressure respectively, while $\Delta p = p - \bar{p}_o$ and $\Delta T = T - \bar{T}_o.$ are the variations in pressure and temperature. These variations are usually small in environmental problems. We define the thermal expansioin coefficient β by :

$$\beta = -\left.\frac{1}{\bar{\rho}_0}\frac{\partial \rho}{\partial T}\right|_p = \left.\frac{1}{\bar{V}_0}\frac{\partial V}{\partial T}\right|_p \tag{4.1.12}$$

where V denotes the specific volume (volume per unit mass), and the bulk modulus $1/\varepsilon$ by

$$\varepsilon = \left.\frac{1}{\bar{\rho}_0}\frac{\partial \rho}{\partial p}\right|_T = -\left.\frac{1}{\bar{V}_0}\frac{\partial V}{\partial p}\right|_T \tag{4.1.13}$$

$$\rho \cong \bar{\rho}_o(1 - \beta\Delta T + \varepsilon\Delta p) \tag{4.1.14}$$

For liquids

$$O(\beta) \sim 10^{-3}/^o K, \quad O(\varepsilon) \sim 10^{-6}/atm$$

Rewrite Eqn. (4.1.5)

$$\rho\frac{De}{Dt} + \rho q_i\frac{Dq_i}{Dt} = \rho q_i f_i + q_i\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial}{\partial x_i}K\frac{\partial T}{\partial x_i} - p\frac{\partial q_i}{\partial x_i} + \tau_{ij}\frac{\partial q_i}{\partial x_j} \tag{4.1.15}$$

In summary, we have, for a perfect gas

$$\boxed{\rho C_v\frac{DT}{Dt} = -p\frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i}K\frac{\partial T}{\partial x_i} + \tau_{ij}\frac{\partial q_i}{\partial x_j}} \tag{4.1.16}$$

and for an incompressible liquid

$$\boxed{\rho C\frac{DT}{Dt} = \frac{\partial}{\partial x_i}K\frac{\partial T}{\partial x_i} + \tau_{ij}\frac{\partial q_i}{\partial x_j}} \tag{4.1.17}$$

Recall that

$$\Phi = \tau_{ij}\frac{\partial q_i}{\partial x_j} \tag{4.1.18}$$

is the rate of viscous dissipation. Thus the rate of change in internal energy is equal to the sum of rate of pressure working to compress the fluid, viscous dissipation, and heat diffusion. More will be said about the incompressiblilty of liquid later.