Notes on
1.63 J/2.21J Fluid Dynamics
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February 25, 2007

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3.9 Oscillatory Boundary Layers

3.9.1 Stokes problem

Near the solid bottom under a wave there is a boundary layer. Let the outside flow have the tangential velocity $u = \Re U(x)e^{-i\omega t}$. Consider the ratio

$$\frac{uu_x, vu_y}{u_t} = O\left(\frac{U_o}{\omega L}\right)$$

If

$$1 \gg \frac{U_o}{\omega L} \gg \frac{\nu}{\omega L^2}$$

then, to a high degree of acuracy in δ^2/L^2, we have, in physical variables:

$$u_x + v_y = 0 \tag{3.9.1}$$

$$u_t + (uu_x + vu_y) = U_t + UU_x + \nu u_{yy} \tag{3.9.2}$$

Keep in mind that the quadratic terms are one order in $U_o/\omega L$ smaller than the linear terms.

Let us introduce, artifically for the sake of keeping track of the small terms the ordering parameter

$$\epsilon = \frac{U_o}{\omega L} \ll 1$$

and insert it in front of the nonlinear terms so that (3.9.2) becomes

$$u_t + \epsilon(uu_x + vu_y) = U_t + \epsilon UU_x + \nu u_{yy} \tag{3.9.3}$$

Let us apply the perturbation method by assuming an approximate solution in the form of a series,

$$u = u_1 + \epsilon u_2 + O(\epsilon^2), \tag{3.9.4}$$

Substituting (3.9.4) into the boundary layer equations and separate terms of different orders, we get perturbation equations of different orders: $O(\epsilon)^0$ and $O(\epsilon)$, etc. After this mission is accopmplished, the order parameter will be discarded.

From the leading order $O(U_o/\omega L)^0$,

$$\frac{\partial u_1}{\partial t} = \frac{\partial U}{\partial t} + \nu \frac{\partial^2 u_1}{\partial y^2} \qquad y > 0 \tag{3.9.5}$$

subject to the boundary conditions that

$$u_1 \to U(x,t) \qquad y \to \infty \tag{3.9.6}$$

and

$$u_1 = 0 \qquad y = 0 \tag{3.9.7}$$

From here on we only consider simple harmonic time dependence, i.e., the tangential inviscid velocity at the outer edge of the boundary layer is

$$U(x,t) = \Re\left(\mathcal{U}(x)e^{-\omega t}\right).$$

Let

$$u_1 = \Re\left[\hat{u}_1(x,y)e^{-i\omega t} + \mathcal{U}e^{-i\omega t}\right] \tag{3.9.8}$$

then

$$-i\omega\mathcal{U} - i\omega\hat{u}_1 = -i\omega\mathcal{U} + \nu\frac{d^2\hat{u}_1}{dy^2}$$

Therefore,

$$\frac{d^2\hat{u}_1}{dy^2} + \frac{i\omega}{\nu}\hat{u}_1 = 0 \tag{3.9.9}$$

$$\hat{u}_1 \to 0, \qquad y \to \infty \tag{3.9.10}$$

$$\hat{u}_1 = -\mathcal{U}(x), \qquad y = 0 \tag{3.9.11}$$

The solution (due to Stokes) is

$$\hat{u}_1 = -\mathcal{U}(x)\exp\left[-(1-i)y\sqrt{\frac{\omega}{2\nu}}\right] \tag{3.9.12}$$

or,

$$u_1 = \Re\left\{\mathcal{U}(x)\left[1 - \exp\left(-(1-i)y\sqrt{\frac{\omega}{2\nu}}\right)\right]e^{-i\omega t}\right\} \tag{3.9.13}$$

The sign of $\sqrt{-i}$ is chosen so that (3.9.10) is satisfied. The boundary layer thickness is

$$\delta = \sqrt{\frac{2\nu}{\omega}} \tag{3.9.14}$$

To complete the leading order solution we calculate the transverse velocity component v_1 in the boundary layer. By continuity:

$$\begin{aligned} v_1 &= -\int_0^y \frac{\partial u_1}{\partial x}dy = ie^{-i\omega t}\frac{d\mathcal{U}}{dx}\int_0^y \left[1 - e^{-(1-i)y/\delta}\right]dy \\ &= -e^{-i\omega t}\frac{d\mathcal{U}}{dx}\left\{y - \frac{\delta}{1-i}\left[1 - e^{-(1-i)y/\delta}\right]\right\} \end{aligned} \tag{3.9.15}$$

which is valid in $y \le O(\delta)$ only. Thus if the inviscid outer flow has tangential variation $\frac{d\mathcal{U}}{dx} \neq 0$, then there is transverse flow v_1 in the boundary layer.

3.9.2 Induced Streaming

The second-order perturbation equation is :

$$\begin{aligned}\frac{\partial u_2}{\partial t} - \nu\frac{\partial^2 u_2}{\partial y^2} &= U\frac{\partial U}{\partial x} - \left(u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y}\right)\\ &= U\frac{\partial U}{\partial x} - \left[\frac{\partial\,(u_1 u_1)}{\partial x} + \frac{\partial\,(u_1 v_1)}{\partial y}\right]\end{aligned}$$

Since U, u_1 and v_1 are simple harmonic in time, the solution for u_2 must have zeroth and second harmonics. Focussing on the zeroth harmonic by taking the average over a period

$$-\nu\frac{\partial^2 \bar{u}_2}{\partial y^2} = \overline{U\frac{\partial U}{\partial x}} - \left(\frac{\partial\overline{u_1 u_1}}{\partial x} + \frac{\partial\overline{u_1 v_1}}{\partial y}\right)$$

On the right-hand-side the last two terms $\overline{u_1 u_1}$, $\overline{u_1 v_1}$ are wave-induced Reynolds stresses. In particular $\rho\overline{u_1 u_1}$ is the rate of transporting x-momentum in the x-directionm, and $\rho\overline{u_1 v_1}$ is the rate of transporting x-momentum in y-direction.

Alternatively:

$$-\nu\frac{\partial^2 \bar{u}_2}{\partial y^2} = \frac{1}{2}\frac{\partial}{\partial x}\overline{U^2} - \frac{1}{2}\frac{\partial}{\partial x}\overline{u_1^2} - \overline{v_1\frac{\partial u_1}{\partial y}}$$

Let

$$\alpha = (1-i)/\delta \tag{3.9.16}$$

Since

$$v_1 = ie^{-i\omega t}\frac{1}{\alpha}\frac{d\mathcal{U}}{dx}\left(\alpha y - 1 + e^{-\alpha y}\right)$$

$$\frac{\partial u_1}{\partial y} = \alpha\mathcal{U}(x)e^{-i\omega t}e^{-\alpha y}$$

Exercise: Show that if

$$a(t) = \Re(Ae^{-i\omega t}), \quad \text{and} \quad b(t) = \Re(Be^{-i\omega t})$$

then the time average of their product is

$$\overline{ab} = \frac{1}{2}\Re(AB^*) = \frac{1}{2}\Re(A^*B). \tag{3.9.17}$$

Using this formula it can be shown that

$$\overline{-v_1\frac{\partial u_1}{\partial y}} = \frac{1}{2}Re\left[\mathcal{U}^*\frac{d\mathcal{U}}{dx}\frac{\alpha^*}{\alpha}e^{-\alpha^* y}\left(\alpha y - 1 + e^{-\alpha y}\right)\right]$$

Thus

$$
\begin{aligned}
-\nu\frac{\partial^2\bar{u}_2}{\partial y^2} &= G(y) \equiv \frac{1}{2}\frac{d|\mathcal{U}|^2}{dx}\left[1-\left(1-e^{-\alpha y}\right)\left(1-e^{-\alpha^* y}\right)\right] \\
&\quad + Re\,\mathcal{U}^*\frac{d\mathcal{U}}{dx}\frac{\alpha *}{\alpha}e^{-\alpha^* y}\left(\alpha y - 1 + e^{-\alpha y}\right) \\
\nu\frac{\partial\bar{u}_2}{\partial y} &= \int_y^\infty G(y')dy' \\
\nu\,\bar{u}_2 &= \int_0^y dy'' \int_{y''}^\infty G(y')dy' \\
&= -y\int_y^\infty G(y')dy' + \int_0^y y'' G(y'')dy''
\end{aligned}
$$

One more integration gives

$$
\begin{aligned}
-\omega\bar{u}_2 &= Re\,F\mathcal{U}\frac{d\mathcal{U}^*}{dx} \\
\text{where}\quad F &= -\frac{1}{2}(1-3i)e^{-(1-i)\eta} - \frac{i}{2}e^{-(1+i)\eta} - \frac{1}{4}(1+i)e^{-2\eta} \\
&\quad + \frac{1}{2}(1+i)\eta e^{-(1-i)\eta} + \frac{3}{4}(1-i)
\end{aligned}
$$

Note that as $y \to \infty$, just outside the boundary layer,

$$\bar{u}_2 = -\frac{3}{4\omega}Re\left[(1-i)\,\mathcal{U}\frac{d\mathcal{U}^*}{dx}\right] \tag{3.9.18}$$

By Taylor expansion we can show that for $\eta \ll 1$,

$$\bar{u}_2 \approx Re\left[\frac{(1+i)}{\omega}\frac{\eta}{2}\mathcal{U}\frac{d\mathcal{U}^*}{dx}\right] \tag{3.9.19}$$

Example : Surface gravity waves

On the free surface of water of contant depth h, let the vertical displacement be

$$\zeta = \Re\left[A\left(e^{ikx} + Re^{-ikx}\right)e^{-i\omega t}\right] \tag{3.9.20}$$

where R denotes the reflection coefficient. The frequency ω frequency and and the wavenumber k are related by

$$\omega^2 = gk\tanh kh \tag{3.9.21}$$

The corresponding velocity potential is

$$\Phi = \Re\left[-\frac{igA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\left(e^{ikx} + Re^{-ikx}\right)e^{-i\omega t}\right] \tag{3.9.22}$$

The inviscid horizontal velocty just above the bed boundary layer is

$$\frac{\partial}{\partial x}\Phi(x,-h,t) = \frac{gkA}{\omega\cosh kh}\Re\left[\left(e^{ikx} - Re^{-ikx}\right)e^{-i\omega t}\right] \tag{3.9.23}$$

We can then identify

$$\mathcal{U} = \frac{gkA}{\omega\cosh kh}\left(e^{ikx} - Re^{-ikx}\right) \tag{3.9.24}$$

For purely progressive waves, $R = 0$

$$\mathcal{U} = \frac{gkA}{\omega\cosh kh}e^{ikx} \tag{3.9.25}$$

hence

$$\frac{d\mathcal{U}^*}{dx} = -ik\frac{gkA}{\omega\cosh kh}e^{-ikx} \tag{3.9.26}$$

The induced streaming velocity is,

$$\bar{u}_2(\infty) = \frac{3}{4\omega}k\left(\frac{gkA}{\omega\cosh kh}\right)^2 \tag{3.9.27}$$

at the upper edge of the boundary layer, and

$$\bar{u}_2(\eta) \approx \frac{\eta}{2\omega}k\left(\frac{gkA}{\omega\cosh kh}\right)^2, \quad \eta \ll 1. \tag{3.9.28}$$

near the bottom of the boundary layer. The velocity profile is monotonic in height.

For purely standing waves $R = 1$, we have

$$\mathcal{U} = \frac{gkA}{\omega\cosh kh}2i\sin kx \tag{3.9.29}$$

and

$$\frac{d\mathcal{U}^*}{dx} = -2ik\frac{gkA}{\omega\cosh kh}\cos kx \tag{3.9.30}$$

Hence

$$\mathcal{U}\frac{d\mathcal{U}^*}{dx} = \left(\frac{gkA}{\omega\cosh kh}\right)^2 2k\sin 2kx \tag{3.9.31}$$

It follows that

$$\bar{u}_2(\infty) = -\frac{3}{4\omega}k\left(\frac{gkA}{\omega\cosh kh}\right)^2\sin 2kx \tag{3.9.32}$$

and

$$\bar{u}_2(\eta) \approx \frac{\eta}{2\omega}k\left(\frac{gkA}{\omega\cosh kh}\right)^2\sin 2kx, \quad \eta \ll 1. \tag{3.9.33}$$

Thus near the bottom of the boundary layer, the streaming velocty converges toward points beneath the amplitude minima. Near the top, the opposite is true. See Figure (3.9.2).

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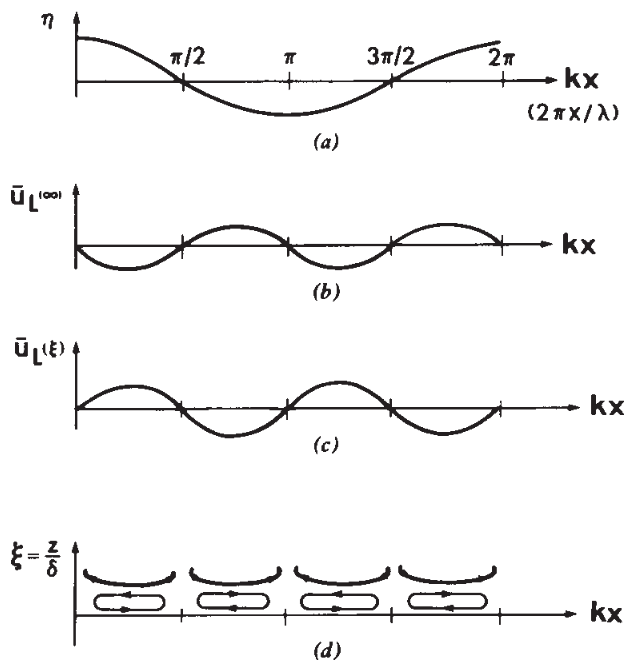

Figure 3.2 Schematic variation of mass transport velocity beneath a standing wave.

3.9.3 Physics of the Induced Streaming

Take progressive water waves as an example: We have outside the boundary layer,

$$u_\infty = U_o \cos(\omega t - kx) \tag{3.9.34}$$

and inside the boundary layer,

$$u = U_o \left[\cos(\omega t - kx) - e^{-y/\delta} \cos(\omega t - kx - y/\delta)\right] \tag{3.9.35}$$

where the velocity amplitude U_o is related to the surface amplitude A by

$$U_o = \frac{gkA}{\omega \cosh kh} = \frac{A\omega}{\sinh kh} \tag{3.9.36}$$

Let us find the induced transverse velocity v

$$\frac{\partial u}{\partial x} = U_o \sin(\omega t - kx) - U_o e^{-y/\delta} \sin(\omega t - kx - y/\delta)$$

$$v_\infty = -\int_0^{y \gg \delta} \frac{\partial u}{\partial x}\, dy = -y\, U_o \sin(\omega t - kx) - \frac{1}{2} U_o k\delta \cos(\omega t - kx) + \frac{1}{2} U_o k\delta \sin(\omega t - kx)$$

Now

$$\overline{u_\infty v_\infty} = -\frac{1}{4} U_o^2 k\delta < 0$$

where the $\sin(\omega t - kx)$ terms in v_∞ are out of phase with u_∞ by $\pi/2$, hence does not contribute to the mean.

Now consider a slice of boundary layer one wavelength long. Because of periodicity, there is no net transfer of momentum or forces at two ends x_0 and $x_0 + 2\pi/k$. But the momentum transfer downwards is $\frac{U_o^2}{4} k\delta$, causing a positive shear stress. To balance it there must be a non-zero $\mu \frac{\partial \bar{u}}{\partial y}$ at all levels y below the top. Hence, the induced streaming velocity is created and $\bar{u} \neq 0$.

The induced streaming in a wave boundary layer is a mechanism for the generation of sand bars, as shown in the following phtographs.

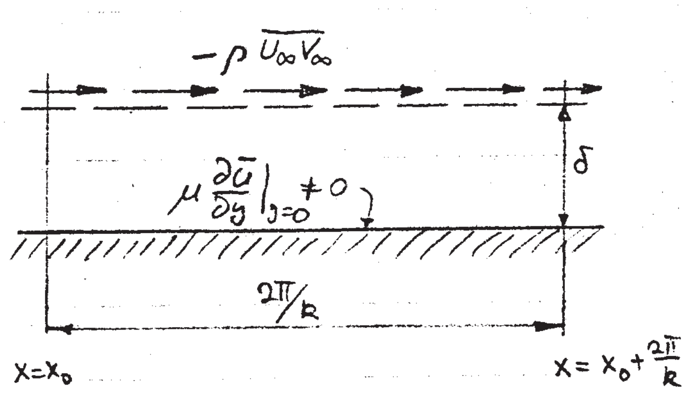

Figure 3.9.1: Reynolds stress and Induced streaming in Stokes layer

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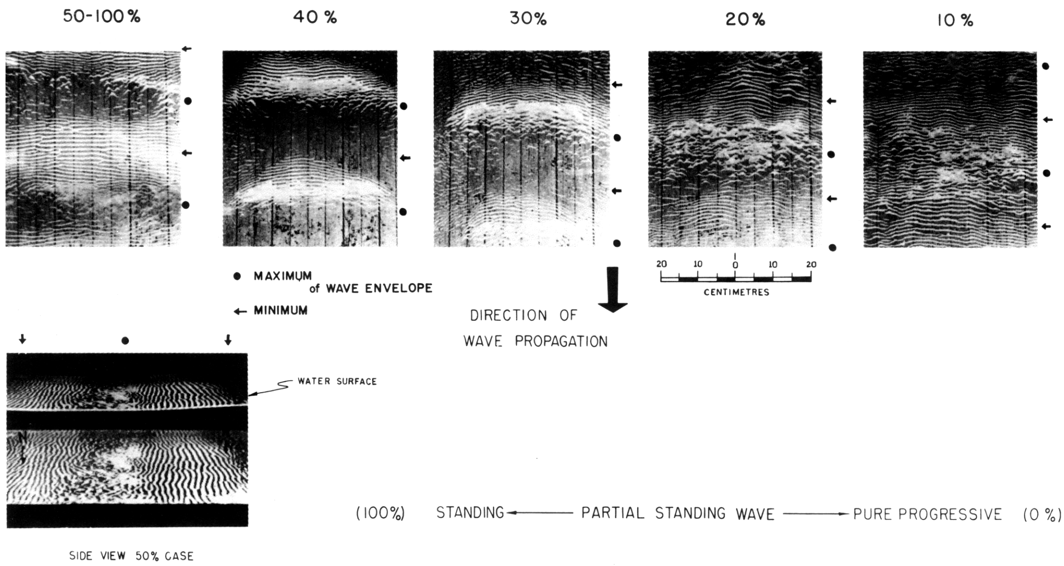

Figure 3.9.2: Accumulation of heavy sand on the metalic bottom of a wave tank. Refection coefficients vary from low to high. Note that for high reflection, heavy sand pile up beneath the wave nodes.