Notes on
1.63 J/2.21J Fluid Dynamics
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3-8impulsive.tex

3.8 Gust, or impulsive flow past a blunt body

Ref: H. Schlichting, Boundary layer theory, p 400 ff.

As an example of unsteady boundary layer, let us consider the initial stage ($U_oT/L \ll 1$) of a boundary layer due to the impulsive start of flow near a blunt body, see the sketch in Figure 3.8.1.

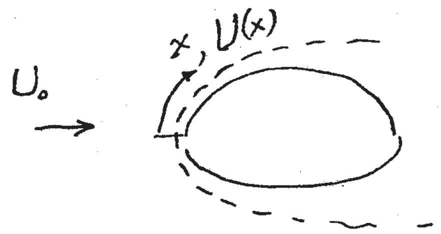

Figure 3.8.1: Boundary layer around a blunt body

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Let us start with the boundary layer approximation and introduce a perturbation expansion in powers of the small ratio U_oT/L,

$$u = u^{(1)} + \left(\frac{U_oT}{L}\right) u^{(2)} + \left(\frac{U_oT}{L}\right)^2 u^{(3)} \cdots, \tag{3.8.1}$$

$$p = p^{(1)} + \left(\frac{U_oT}{L}\right)p^{(2)} + (\frac{U_oT}{L}\right)^2 p^{(3)} + \cdots \tag{3.8.2}$$

We then get

$$u_x^{(1)} + v_y^{(1)} + \left(\frac{U_oT}{L}\right)\left(u_x^{(2)} + v_y^{(2)}\right) + \cdots = 0, \tag{3.8.3}$$

and

$$u_t^{(1)} + \frac{U_oT}{L} u_t^{(2)} + \frac{U_oT}{L}(u^{(1)}u_x^{(1)} + v^{(1)}u_y^{(1)}) + O\left(\frac{U_oT}{L}\right)^2$$
$$= \frac{U_oT}{L} UU_x + u_{yy}^{(1)} + \frac{U_oT}{L} u_{yy}^{(2)} + O\left(\frac{U_oT}{L}\right)^2 \tag{3.8.4}$$
(3.8.5)

Equating the coefficients of $\left(\frac{U_oT}{L}\right)^0$ we get the first (leading) order perturbation equations in normalized coordinates,

$$u_x^{(1)} + v_y^{(1)} = 0, \tag{3.8.6}$$

$$u_t^{(1)} = u_{yy}^{(1)} \tag{3.8.7}$$

subject to the initial conditions:

$$u^{(1)} = v^{(1)} = 0. \quad t = 0, \quad \forall y; \tag{3.8.8}$$

and the boundary condtions

$$u^{(1)} = v^{(1)} = 0. \quad y = 0, \quad \forall t; \tag{3.8.9}$$

$$u^{(1)} = U, \quad y \to \infty \tag{3.8.10}$$

Equating the coefficient of $\left(\frac{U_oT}{L}\right)$, we get the second order perturbation equations in normalized coordinates,

$$u_x^{(2)} + v_y^{(2)} = 0, \tag{3.8.11}$$

$$u_t^{(2)} + (u^{(1)}u_x^{(1)} + v^{(1)}u_y^{(1)}) = UU_x + u_{yy}^{(2)} + O\left(\frac{U_oT}{L}\right)^2 \tag{3.8.12}$$

subject to the same initial and boundary conditions on the wall as the first order problem, except that

$$u^{(2)} \to 0, \quad y \to \infty \tag{3.8.13}$$

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To return to physical variables, we need only add the coeficient ν in front of the viscous stress term u_{yy} in (3.8.7), and (3.8.12). The first order problem for the tangential velocity is precisely the Rayleigh problem

$$u_t^{(1)} = u_{yy}^{(1)} \tag{3.8.14}$$

subject to the initial conditions:

$$u^{(1)} = 0. \quad t = 0, \quad \forall y; \tag{3.8.15}$$

and the boundary condtions

$$u^{(1)} = 0. \quad y = 0, \quad \forall t; \tag{3.8.16}$$

$$u^{(1)} = U, \quad y \to \infty \tag{3.8.17}$$

The solution is

$$u^{(1)}(x,y,t) = U(x)\text{erf}(\eta) = U(x)\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\,d\eta \tag{3.8.18}$$

where

$$\eta = \frac{y}{\sqrt{2\nu t}} \tag{3.8.19}$$

Integrating the continuity equation (3.8.6) we get

$$v^{(1)} = -\int_0^y \frac{\partial u_1}{\partial x}dy = -\frac{dU}{dx}2\sqrt{\nu t}\int_0^\eta \text{erf}(\eta)\,d\eta \tag{3.8.20}$$

To simply the notation we introduce

$$\text{erf}(\eta) = \zeta_0'(\eta), \qquad \int_0^\eta \text{erf}(\eta)\,d\eta = \zeta_0(\eta) \tag{3.8.21}$$

so that

$$u^{(1)} = U(x)\zeta_0'(\eta), \quad v^{(1)} = -\frac{dU}{dx}2\sqrt{\nu t}\zeta_0(\eta) \tag{3.8.22}$$

The second-order approximation is

$$u_t^{(2)} - \nu u_{yy}^{(2)} = UU_x - u^{(1)}u_x^{(1)} - v^{(1)}u_y^{(1)} \tag{3.8.23}$$

subject to the initial and boundary conditions that

$$u^{(2)}(y,0) = 0, \quad u^{(2)}(y,t) = 0 \quad \text{for} y = 0, \infty \tag{3.8.24}$$

The right hand side of (3.8.23) can be worked out so that

$$\begin{aligned} u_t^{(2)} - \nu u_{yy}^{(2)} &= UU_x\left[1 - (\text{erf}(\eta))^2 + e^{-\eta^2}\int_0^\eta \text{erf}(\eta)\,d\eta\right] \\ &= UU_x\left[1 - (h')^2 + hh''\right] = UU_xF(\eta) \end{aligned} \tag{3.8.25}$$

A similarity solution is possible. Let us seek a one-parameter transformation,

$$u^{(2)} = \lambda^a u^{(2)'}, \quad t = \lambda^b t', \quad y = \lambda^c y'$$

From (3.8.23) we get

$$\lambda^{a-b}\frac{\partial u^{(2)'}}{\partial t'} - \nu\lambda^{a-2c}\frac{\partial^2 u^{(2)'}}{\partial y'^2} = UU_xF(\lambda^{c-b/2}\eta')$$

Note that x is just a parameter. Clearly $a = b = 2c$ so that we can take

$$\frac{u^{(2)}}{t} = f(\eta)UU_x \tag{3.8.26}$$

Substituting (3.8.26) into (3.8.25), we get a linear ordinary differential equation

$$f'' + 2\eta f' - 4f = 4\left[(\zeta_0')^2 - \zeta_0\zeta_0'' - 1\right] \tag{3.8.27}$$

subject to the boundary conditions that

$$f = 0, \quad \eta = 0, \infty \tag{3.8.28}$$

The solution is not difficult (see Schlichting, eq. 15.43, p. 400).

$$\begin{aligned} f &= \operatorname{erfc}(\eta)\left[-\frac{3}{\sqrt{\pi}}e^{-\eta^2} + 2 - \left(\frac{3}{\sqrt{\pi}} + \frac{4}{3\pi\sqrt{\pi}}\right) + \frac{\sqrt{\pi}}{2}(2\eta^2+1)\right] \\ &+ \frac{1}{2}(2\eta^2-1)\operatorname{erfc}^2(\eta) + \frac{2}{3}e^{-2\eta^2} \\ &+ e^{-\eta^2}\left[\frac{\eta}{\sqrt{\pi}} - \frac{4}{3\pi} + \eta\left(\frac{3}{\sqrt{\pi}} + \frac{4}{3\pi\sqrt{\pi}}\right)\right] \end{aligned} \tag{3.8.29}$$

The solution is plotted in Figure 3.8.2.

The total solution is

$$u = U\operatorname{erf}(\eta) + tUU_x f(\eta) \tag{3.8.30}$$

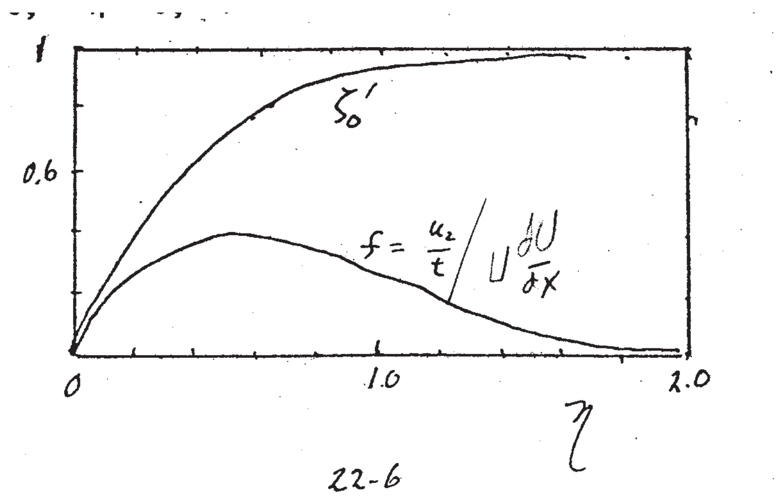

Figure 3.8.2: Solution to the problem of impulsive start.

Separation

For a given $U(x)$ when and where will separation first occur? Namely, when is

$$\frac{\partial u}{\partial y} = 0 at y = 0$$

Let us use (3.8.30) for a crude estimate. Since

$$\frac{\partial u}{\partial y} = [U(\mathrm{erf}\eta)' + UU_x\, t\, f'(\eta)]\, \frac{\partial \eta}{\partial y}$$

It can be show n that at $\eta = 0$,

$$(\mathrm{erf}\eta)' = \frac{2}{\sqrt{\pi}}, \quad f'(\eta) = \frac{2}{\sqrt{\pi}}\left(1 + \frac{4}{3\pi}\right)$$

It follows that

$$U + t_s\left(1 + \frac{4}{3\pi}\right) UU_x = 0$$

or

$$t_s = -\frac{0.7}{UU_x} \tag{3.8.31}$$

Note that $t_s > 0$ only for $U_x < 0$, i.e., a decelerated flow. This is a very crude and mathematically illigitimate estimate since we are equating two terms of different order.

Neveltherless let us apply this result to the impulsive flow passing a circular cylinder from the left. Let U_o be the constant velocity at infinity and the polar angle θ be measured from the upstream stagnation point, then $x = a\theta$ where a is the radius, see Figure 3.8.3. It is well known in the potential theory that the potential is

$$\phi = U_o\left(r + \frac{a^2}{r}\right)\cos(\pi - \theta)$$

The tangential velocity along the cylinder $r = a$ is

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{U_o}{r}\left(r + \frac{a^2}{r}\right)\sin(\pi - \theta), \quad r = a$$

or

$$U = 2U_o \sin(\pi - \theta) = 2U_o \sin(\theta) = 2U_o \sin x/a$$

The minimum t_s occurs at the riear stagnation point, $x = \pi a$ at which

$$t_s = \frac{0.35a}{U_o}, \quad \text{or} \quad \frac{U_o t_s}{a} = 0.35$$

Note that the last condition indicates the illigitimacy of this estimate. Nevertheless we use it here as an order-of-magnitude guide which may be improved by working out higher order terms.

In offshore stuctures, wave induced oscillatory flows acound a pile can be separated and hence affect the drag force on the pile. As an order estimate we take $U_o = \omega A$ where ω =frequency and A =wave amplitude. Hence there is no separation if

$$\frac{\omega A t_s}{a} < 0.35, \quad \text{or} \quad \frac{A}{a} < \frac{0.35}{\omega t_s}$$

Since flow changes direction after every half period π/ω, there is no separation in every half period if

$$\frac{A}{a} < \frac{0.35}{\pi} = 0.1$$

This is of course very crude. Experimentally Keulegan and Carpenter have estiblished that separation occurs in waves if A/a exceeds 1. The ratio A/a is now known as the Keulegan and Carpenter number.

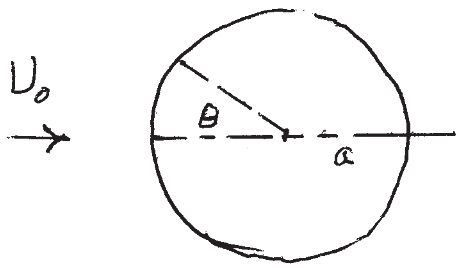

Figure 3.8.3: Definition of coordinates for a circular cylinder.

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Appendix: Details of perturbation analysis

Consider

$$1 \gg \frac{\nu}{\omega L^2} \gg \epsilon = \frac{U_o T}{L} \sim \frac{U_o}{\omega L}$$

Dimensionless equations

$$u_x + v_y = 0, \tag{3.8.1}$$

$$u_t + \epsilon(uu_x + vu_y) = U_t + \epsilon U U_x + u_{yy} \tag{3.8.2}$$

Introduce

$$u = u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots, \quad v = v_1 + \epsilon v_2 + \epsilon^2 v_3 + \cdots, \tag{3.8.3}$$

Plugging into (3.8.1),

$$\frac{\partial}{\partial x}\left(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots,\right) + \frac{\partial}{\partial y}\left(v_1 + \epsilon v_2 + \epsilon^2 v_3 + \cdots,\right) = 0$$

Plugging into (3.8.2)

$$\frac{\partial}{\partial t}\left(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots,\right)$$

$$+\epsilon(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots)\frac{\partial}{\partial x}(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots)$$

$$+\epsilon(v_1 + \epsilon v_2 + \epsilon^2 v_3 + \cdots)\frac{\partial}{\partial y}(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots)$$

$$= \frac{\partial U}{\partial t} + \epsilon U\frac{\partial U}{\partial x} + \frac{\partial^2}{\partial y^2}(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots)$$

Order $O(\epsilon^0)$:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{3.8.4}$$

$$\frac{\partial u_1}{\partial t} = \frac{\partial U}{\partial t} + \frac{\partial^2 u_1}{\partial y^2} \tag{3.8.5}$$

Order $O(\epsilon)$:

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{3.8.6}$$

$$\frac{\partial u_2}{\partial t} + \left(u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y}\right) = U\frac{\partial U}{\partial x} + \frac{\partial^2 u_2}{\partial y^2} \tag{3.8.7}$$

Boundary conditions: $O(\epsilon^0)$:

$$u_1 = v_1 = 0, \quad y = 0 \tag{3.8.8}$$

$$u_1 \to U, \quad y \to \infty \tag{3.8.9}$$

$O(\epsilon)$:

$$u_2 = v_2 = 0, \quad y = 0 \tag{3.8.10}$$