Notes on
1.63 J/2.21J Fluid Dynamics
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3.6 Transient boundary layer along a flat plate

Consider the two dimensional boundary layer near edge of half infinite plane along the x axis due to the impulsive start along its own plane. There is no motion anywhere for $t < 0$. At $t = 0$ the plane suddenly advances from right to left perpendicular to its leading edge. What is the boundary layer flow? Referring to Figure 3.6.1 where the coordinate system is

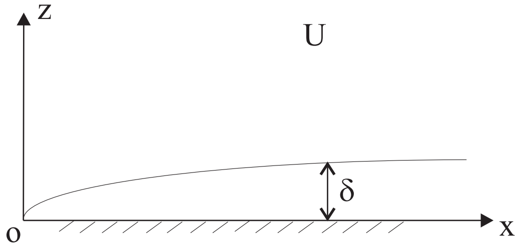

Figure 3.6.1: Front of boundary layer

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fixed on the plane. For all $t > 0$,

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial x} = 0$$

The boundary layer equation reads:

$$\rho(u_t + uu_x + wu_z) = \mu u_{zz} \tag{3.6.1}$$

and the Karman momentum inetgral equation reads

$$\frac{\partial}{\partial t}\int_0^\infty \rho(U-u)dz + \frac{\partial}{\partial x}\int_0^\infty \rho\left(Uu - u^2\right)dz = \mu \left.\frac{\partial u}{\partial z}\right|_0 \tag{3.6.2}$$

Let us assume

$$\frac{u}{U} = f(\eta), \qquad \eta \equiv \frac{z}{\delta(x,t)} \tag{3.6.3}$$

then

$$\delta_1 = \delta\int_0^\infty (1 - f(\eta))d\eta = \alpha_1\delta \tag{3.6.4}$$

$$\delta_2 = \delta \int_0^\infty f(\eta)(1 - f(\eta))d\eta = \alpha_2 \delta \tag{3.6.5}$$

and

$$u_z = \frac{U}{\delta} f'(0). \tag{3.6.6}$$

Substituting into Eq. (3.6.2)

$$\alpha_1 U \frac{\partial \delta}{\partial t} + \alpha_2 U^2 \frac{\partial \delta}{\partial x} = \nu \frac{U}{\delta} f'(0)$$

Therefore,

$$(\alpha_1) \frac{\partial\left(\delta^2/2\right)}{\partial t} + (\alpha_2 U) \frac{\partial\left(\delta^2/2\right)}{\partial x} = \nu f'(0) \tag{3.6.7}$$

which is a first order wave (hyperbolic) equation for $\delta^2/2$. We must add the boundary and initial conditions:

$$\delta(0,t) = 0, \quad \forall t > 0 \tag{3.6.8}$$

$$\delta(x,0) = 0, \quad \forall x > 0 \tag{3.6.9}$$

3.6.1 Heuristic solution

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The fact that the boundary value in (3.6.8) is independent of t for all $t > 0$ suggests that the solution is independent t for sufficiently large t, i.e., after the initial transient dies out,

$$\frac{\partial}{\partial x}\left(\frac{\delta^2}{2}\right) = \frac{\nu f'(0)}{\alpha_2 U} \tag{3.6.10}$$

subject to the boundary condition (3.6.8). The solution is simply

$$\delta = \sqrt{\frac{2\nu f'(0) x}{\alpha_2 U}} \tag{3.6.11}$$

This is the approximate version of the solution by Blasius who solved the steady boundary layer equation

$$u u_x + v u_y = \nu u_{yy} \quad \text{for} \quad x > 0, \quad y > 0 \tag{3.6.12}$$

by the method of similarity.

On the other hand, the fact that the initial value in (3.6.8) is independent of x for all x suggests that the solution is independent of x for sufficiently large x, i.e.,

$$\frac{\partial}{\partial t}\left(\frac{\delta^2}{2}\right) = \frac{\nu f'(0)}{\alpha_1} \tag{3.6.13}$$

subject to the initial condition (3.6.9). The solution is simply

$$\delta = \sqrt{\frac{2\nu f'(0)t}{\alpha_1}} \tag{3.6.14}$$

This is the approximate version of the solution by Rayleigh's problem which is known to be governed by

$$u_t = \nu u_{yy} \quad \text{for} \quad t > 0, \quad y > 0 \tag{3.6.15}$$

Let us assume that the common borderline of two solutions in the $x - t$ plane is defined by equating the two δ's given by (3.6.11) and (3.6.14),

$$x = \frac{\alpha_2 U t}{\alpha_1} \tag{3.6.16}$$

Thus (3.6.11) holds in the wedge $x > \frac{\alpha_2 U t}{\alpha_1} > 0$ and (3.6.14) holds in the wedge $0 < x < \frac{\alpha_2 U t}{\alpha_1}$. See Figure 3.6.2.

Physically: if $t > \alpha_1 x/\alpha_2 U$, (3.6.11) applies and one has the steady Blasius flow past a semi-infinite plate; the boundary layer is already in the steady state. See Figure 3.6.3, If $t < \alpha_1 x/\alpha_2 U$, (3.6.14) applies and one has Rayleigh's problem of impulsively started plane infinite in both $x < 0$ and $x > 0$. The boundary layer is still in the initial stage and the effect of the leading edge is not felt elsewhere. The result is summarized in Figure **??**.

To calculate α_1 and α_2 let us make a special choice of the velocity profile so that $f(0) = 0, f'(1) = f''(1) = 0$ (due to Karman and Polhausen)

$$f(\eta) = 2\eta - 2\eta^3 + \eta^4, \quad 0 < \eta < 1 \tag{3.6.17}$$

Note that

$$f(0) = 0, \quad f'(\eta) = 2 - 6\eta^2 + 4\eta^3, \quad f''(\eta) = -12\eta + 12\eta \tag{3.6.18}$$

then

$$\begin{aligned} \alpha_1 &= \int_0^1 \left(1 - 2\eta + 2\eta^3 - \eta^4\right) d\eta = 3/10 = 0.3 \\ \alpha_2 &= \int_0^1 \left(2\eta - 2\eta^3 + \eta^4\right)\left(1 - 2\eta + 2\eta^3 - \eta^4\right) d\eta = 37/315 \cong 0.117 \end{aligned}$$

and

$$f'(0) = 2, \quad \text{or} \quad \left.\frac{\partial u}{\partial z}\right|_{z=0} = \frac{2U}{\delta}$$

Hence the displacement thickness of the Blasius boundary layer is

$$\delta_1 = \alpha_1 \delta = \alpha_1 \sqrt{\frac{2\nu f'(0)x}{\alpha_2 U}} = 1.754\sqrt{\frac{\nu x}{U}} \tag{3.6.19}$$

The momentum integral method is the special case of the moment method, since the Karman equation is the zeroth moment of the boundary layer equation. For the classical steady boundary layer problem solved exactly by Blasius using the similarity method, the momentum integral approximation gives fairly good results, even with various crude profiles, see Table 3.1 below.

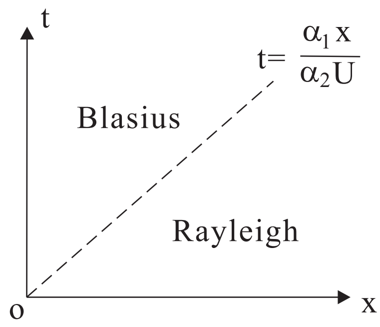

Figure 3.6.2: Evolution of a transient boundary layer. (a) Blasius region and Rayleigh region

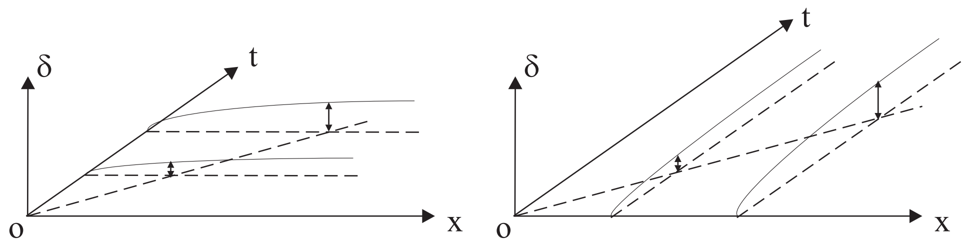

Figure 3.6.3: Evolution of a transient boundary layer. (Left) Blasius region, (Right) Rayleigh region

3.6.2 Formal solution

A more formal solution of (3.6.7) can be obtained as follows: Let

$$a = \alpha_1 \qquad b = \alpha_2 U \qquad c = \nu f'(0)$$

then

$$a\frac{\partial}{\partial t}\left(\frac{\delta^2}{2}\right) + b\frac{\partial}{\partial x}\left(\frac{\delta^2}{2}\right) = c \tag{3.6.20}$$

This is a linear hyperbolic partial differential equation of the first order. The solution can be facilitated by a the method of characteristics. Let us make change of coordinates from (x, t) to the characteristic coordinates (ξ, ζ) where

$$\xi = ax + bt \qquad \zeta = ax - bt \tag{3.6.21}$$

then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial t}; \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial x}$$

$$\begin{aligned}\left(\delta^2\right)_t &= \left(\delta^2\right)_\xi b + \left(\delta^2\right)_\zeta (-b)\\ \left(\delta^2\right)_x &= \left(\delta^2\right)_\xi a + \left(\delta^2\right)_\zeta a\end{aligned}$$

Velocty profile	Displacment boundary layer thickness
$\frac{u}{U} = f(\eta), \quad \eta = \frac{y}{\delta(x)}$	$\delta_1 \sqrt{\frac{U}{\nu x}}$
η	1.732
$\sin \frac{\pi\eta}{2}$	1.741
$2\eta - 2\ \eta^3 + \eta^4$	1.754
Blausius	1.721

Table 3.1: Comparson of approximate solution by momentum integral method with the exact solution of Blasius

Therefore, from (3.6.7),

$$\frac{1}{2}\left[ab\left(\delta^2\right)_\xi - ab\left(\delta^2\right)_\zeta + ab\left(\delta^2\right)_\xi + ab\left(\delta^2\right)_\zeta\right] = c$$

$$\frac{1}{2}\left(\delta^2\right)_\xi = \frac{c}{2ab} = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}$$

Integrate once

$$\frac{\delta^2}{2} = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\xi + G(\zeta) \tag{3.6.22}$$

where G is an arbitrary function of ζ.

To determine $G(\zeta)$ we first use the initial condition that $\delta = 0$ at $t = 0$ <u>for all $x > 0$</u>.

$$0 = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(\xi)_{t=0} + G(\zeta)_{t=0}$$

or

$$0 = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(ax) + G(ax) \quad \text{for} \quad x > 0.$$

Therefore

$$G(\zeta) = -\frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\zeta, \qquad \zeta > 0 \tag{3.6.23}$$

What is $G(\zeta)$ for $\zeta < 0$? Let us use the boundary condition at $x = 0$ that $\delta = 0$ for all $t > 0$.

$$0 = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(\xi)_{x=0} + G(\zeta)_{x=0}$$

or

$$0 = -\frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(-bt) + G(-bt), \qquad t > 0$$

Note that for $t > 0$, $-bt < 0$. Therefore,

$$G(\zeta) = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\zeta \qquad \text{for all} \quad \zeta < 0. \tag{3.6.24}$$

Eqs. (3.6.23) and (3.6.24) complete the solution for all $\zeta > 0$ and $\zeta < 0$.

Let us return to x and t

$$\begin{aligned}\frac{\delta^2}{2} &= \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\xi - \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\zeta = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}2bt \qquad \text{for} \quad ax - bt > 0 \\ &= \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\xi + \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}\zeta = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}2ax \qquad \text{for} \quad ax - bt < 0\end{aligned}$$

Therefore,

$$\delta = \sqrt{\frac{2\nu f'(0)}{\alpha_1}t} \qquad \text{for} \quad x > \frac{\alpha_2 U}{\alpha_1}t \tag{3.6.25}$$

and

$$\delta = \sqrt{\frac{2\nu f'(0)}{\alpha_2 U}x} \qquad \text{for} \quad x < \frac{\alpha_2 U}{\alpha_1}t \tag{3.6.26}$$

These are the same as (3.6.14) and (3.6.11).