Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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3-10pert.tex

3.10 Details of perturbation analysis

Consider

$$1 \gg \frac{\nu}{\omega L^2} \gg \epsilon = \frac{U_o T}{L} \sim \frac{U_o}{\omega L}$$

Dimensionless equations

$$u_x + v_y = 0, \tag{3.10.1}$$

$$u_t + \epsilon(u u_x + v u_y) = U_t + \epsilon U U_x + u_{yy} \tag{3.10.2}$$

Introduce

$$u = u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots, \quad v = v_1 + \epsilon v_2 + \epsilon^2 v_3 + \cdots, \tag{3.10.3}$$

Plugging into (3.10.1),

$$\frac{\partial}{\partial x}\left(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots,\right) + \frac{\partial}{\partial y}\left(v_1 + \epsilon v_2 + \epsilon^2 v_3 + \cdots,\right) = 0$$

Plugging into (3.10.2)

$$\frac{\partial}{\partial t}\left(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots,\right)$$

$$+\epsilon(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots)\frac{\partial}{\partial x}(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots)$$

$$+\epsilon(v_1 + \epsilon v_2 + \epsilon^2 v_3 + \cdots)\frac{\partial}{\partial y}(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots)$$

$$= \frac{\partial U}{\partial t} + \epsilon U \frac{\partial U}{\partial x} + \frac{\partial^2}{\partial y^2}(u_1 + \epsilon u_2 + \epsilon^2 u_3 + \cdots)$$

Order $O(\epsilon^0)$:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{3.10.4}$$

$$\frac{\partial u_1}{\partial t} = \frac{\partial U}{\partial t} + \frac{\partial^2 u_1}{\partial y^2} \tag{3.10.5}$$

Order $O(\epsilon)$:

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{3.10.6}$$

$$\frac{\partial u_2}{\partial t} + \left(u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y}\right) = U \frac{\partial U}{\partial x} + \frac{\partial^2 u_2}{\partial y^2} \tag{3.10.7}$$

Boundary conditions: $O(\epsilon^0)$:

$$u_1 = v_1 = 0, \quad y = 0 \tag{3.10.8}$$

$$u_1 \to U, \quad y \to \infty \tag{3.10.9}$$

$O(\epsilon)$:

$$u_2 = v_2 = 0, \quad y = 0 \tag{3.10.10}$$