Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
by Chiang C. Mei, MIT
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2-4-apile.tex
References:

Landau & Lifshitz, *Fluid Mechanics*, 1959.
H. Huppert : (JFM) (For modelling lava flow.)

2.4.a. Release of a fluid pile from a horizontal plane

On a horizontal plane $\theta = 0$. Eq (2.2.22) reduces to

$$\frac{\partial h}{\partial t} = \alpha \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) \tag{2.4.a.1}$$

where $\alpha = \rho g/3\mu$. This is a nonlinear diffusion equation where the diffusivity ah^3 increases with the unknown h.

Consider the release of a fluid pile with the initial volume Q confined in a small region. Note that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} h(x,t)\,dx = \alpha\left[h^3\frac{\partial h}{\partial x}\right]_{-\infty}^{\infty} = 0$$

Thus

$$\int_{-\infty}^{\infty} h(x,t)\,dx = \int_{-\infty}^{\infty} h(x,0)\,dx = Q \tag{2.4.a.2}$$

i.e., the total mass is constant. The initial-value problem governed by (2.4.a.1) and (2.4.a.2) can be solved by the method of similarity, which reduces the PDE to an ODE (Landau & Lifshitz).

2.4.a.1. The similarity method

Consider the one-parameter transformation:

$$h = \lambda^a h', \quad x = \lambda^b x', \quad t = \lambda^c t' \tag{2.4.a.3}$$

where (a, b, c) will be chosen to leave the initial value problem (PDE and initial condition) unchanged (invariant). Substituting (2.4.a.3) into (2.4.a.1),

$$\lambda^{a-c}\frac{\partial h'}{\partial t'} = \lambda^{4a-2b}\alpha\frac{\partial}{\partial x'}\left(h'^3\frac{\partial h'}{\partial x'}\right) \tag{2.4.a.4}$$

and then in (2.4.a.2),

$$\lambda^{a+b}\int_{-\infty}^{\infty} h'(x,0)\,dx' = Q \tag{2.4.a.5}$$

we find the conditions for invariance,

$$a-c = 4a-2b; \quad a+b=0; \quad i.e., \quad b=-a, \quad c=-5a; \tag{2.4.a.6}$$

implying also

$$h=\lambda^a h', \quad x=\lambda^{-a}x', \quad t=\lambda^{-5a}t' \tag{2.4.a.7}$$

This suggests the followng combination of variables:

$$\xi = \frac{x}{(At)^{1/5}}, \quad f(\xi) = h(Bt)^{1/5}, \tag{2.4.a.8}$$

ξ is called a *similarity* variable. In the space-time plane, at all points on the curve of constant $\xi = x/(Bt)^{1/5}$, $f = h(Bt)^{1/5}$ is the same. Thus for a fixed ξ, $h(Bt)^{1/5}$ is fixed, implying that $h(Bt)^{1/5}$ is a function of ξ, i.e.,

$$h = \frac{f(\xi)}{(Bt)^{1/5}} \tag{2.4.a.9}$$

Physically the maximum of h decays in time as $t^{-1/5}$, while the pile spreads in time as $t^{1/5}$.

Subsituting (2.4.a.9) into (2.4.a.1), and using the facts,

$$\frac{\partial h}{\partial t} = -\frac{1}{5}\frac{1}{B^{1/5}}\frac{f}{t^{6/5}} - \frac{1}{5}\frac{\xi f'}{B^{1/5}t^{6/5}} \qquad \frac{\partial h}{\partial x} = \frac{1}{B^{1/5}t^{1/5}}\frac{f'}{A^{1/5}t^{1/5}}$$

we find

$$-\frac{1}{5}\frac{1}{B^{1/5}t^{6/5}}\left(f+\xi\frac{df}{d\xi}\right) = \frac{\alpha}{B^{4/5}t^{4/5}A^{2/5}t^{2/5}}\frac{d}{d\xi}\left(f^3\frac{df}{d\xi}\right)$$

From the initial condition, we get

$$\left(\frac{At}{Bt}\right)^{1/5}\int_{-\infty}^{\infty} f(\xi)d\xi = Q$$

For simpliicty let us set

$$A^{2/5}B^{3/5} = \alpha \tag{2.4.a.10}$$

and

$$\left(\frac{A}{B}\right)^{1/5} = Q, \tag{2.4.a.11}$$

We then get an ODE:

$$5\frac{d}{d\xi}\left(f^3\frac{df}{d\xi}\right) + \frac{d}{d\xi}(\xi f) = 0 \tag{2.4.a.12}$$

subject to

$$\int_{-\infty}^{\infty} f(\xi)d\xi = 1. \tag{2.4.a.13}$$

at $t \to 0$. Solving (2.4.a.10) and (2.4.a.11) we get

$$A = \alpha Q^3, \quad B = \frac{\alpha}{Q^2} \tag{2.4.a.14}$$

2.4.a.2. Solution

Integrating (2.4.a.12),

$$5f^3\frac{df}{d\xi}+\xi f=0$$

The integration constant is zero because of symemetry at $x=0$ so that $f'(0)=0$. Now

$$5f^2\frac{df}{d\xi}+\xi=0,\quad\text{implying}\quad\frac{5}{3}df^3+\xi d\xi=0$$

Integrating,

$$f^3=\frac{3}{5}\frac{\xi_0^2-\xi^2}{2},$$

where $f(\xi_0)=0$, we get

$$f=\left[\frac{3}{10}(\xi_0^2-\xi^2)\right]^{1/3},\quad -\xi_0<\xi<\xi_0;\qquad =0,\quad\text{othewise.}\tag{2.4.a.15}$$

To find ξ_0 we use (2.4.a.13),

$$\int_{-\xi_0}^{\xi_0}f(\xi)d\xi=1.$$

or

$$\int_{-\xi}^{\xi_0}\left[\xi_0^2-\xi^2)\right]^{1/3}d\xi=\left(\frac{10}{3}\right)^{1/3}$$

This determines ξ_0. The rest is algebra.

Let $\zeta=(\xi/\xi_0)^2$, then

$$d\zeta=2\xi d\xi/{\xi_0}^2,\quad d\xi=\frac{\xi_0}{2}\frac{d\zeta}{\zeta^{1/2}}$$

$$\int_{-\xi}^{\xi_0}\left[\xi_0^2-\xi^2)\right]^{1/3}d\xi=(\xi_0)^{5/3}\int_0^1(1-\zeta)^{1/3}\zeta^{-1/2}d\zeta=(\xi_0)^{5/3}\frac{2\sqrt{\pi}}{5}\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}$$

where $\Gamma(z)$ is the Gamma function, hence

$$\xi_0=\frac{x_0}{(\alpha Q^3t)^{1/5}}=\left(\frac{10}{3}\right)^{1/5}\left(\frac{5}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}\right)^{3/5}\tag{2.4.a.16}$$

The maximum width is

$$x_0=\left(\frac{10}{3}\right)^{1/5}\left(\frac{5}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}\right)^{3/5}(\alpha Q^3t)^{1/5}\tag{2.4.a.17}$$

The maximum depth is

$$h(0,t)=\frac{f(0)}{(Bt)^{1/5}}=\left[\frac{3\xi_0^2}{10}\right]^{1/3}\left(\frac{Q^2}{\alpha t}\right)^{1/5}\tag{2.4.a.18}$$