Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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1.3 Kinematic transport theorem

In particle mechanics, Newton's second law requires the balance between the rate of change of particle momentum and the applied forces. In continuum mechanics, one needs to calculate the rate of change in a volume of fluid. Now different choices can be made for the control volume, such as a geometric volume fixed in space, or moving in a prescribed manner, or a material volume consisting of the same body of fluid. Calculation of the rate can be facilitated by the so-called kinematic transport theorem. We first derive a theorem for any moving volume $V(t)$ bounded by $S(t)$, see Figure 1.3.1.

Theorem 1 *Let $G(\mathbf{x},t)$ be some fluid property per unit volume, then*

$$\frac{d}{dt}\iiint_V G\,dV = \iiint_V \frac{\partial G}{\partial t}\,dV + \iint_S GU_n\,dS \tag{1.3.1}$$

where U_n is the normal component of the velocity of a point on S.

Note first that

$$\left[\iiint_V G(\mathbf{x},t)\,dV\right]_{t+dt} = \iiint_{V(t+dt)} G(\mathbf{x},t+dt)\,dV$$
$$= \iiint_{V(t+dt)}\left[G(\mathbf{x},t) + \frac{\partial G}{\partial t}dt + O(dt)^2\right]dV$$

Due to the movement of $S(t)$ the volume change for a surface element dS is $U_n\,dt\,dS$, thus

$$\iiint_{V(t+dt)} = \iiint_{V(t)} + \iiint_{\Delta V} = \iiint_{V(t)} + \iint_{S(t)} U_n\,dS\,dt$$

We then have

$$\left[\iiint_V G(\mathbf{x},t)\,dV\right]_{t+dt} = \iiint_{V(t)}\left[G(x,t) + \frac{\partial G}{\partial t}dt + O(dt)^2\right]dV$$

$$+\iint_{S(t)} U_n\,dS\,dt\left[G(x,t) + \frac{\partial G}{\partial t}dt + O(dt)^2\right]dV$$

$$= \iiint_{V(t)} G(x,t)dV + \left[\iiint_{V(t)}\frac{\partial G}{\partial t}dV + \iint_{S(t)} U_n\,dS\,G(x,t)\right]dt + O(dt)^2$$

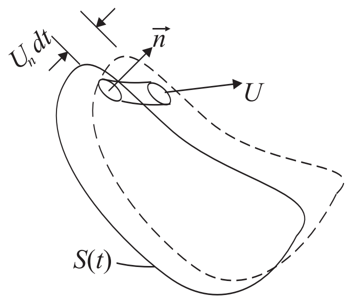

Figure 1.3.1: A moving volume in the fluid

Since

$$\frac{d}{dt}\iiint_V G\,dV = \lim_{dt\to 0}\frac{1}{dt}\left\{\left[\iiint_V G(\mathbf{x},t)\,dV\right]_{t+dt} - \iiint_{V(t)} G(x,t)dt\right\}$$

(1.3.1) is proven.

If $V(t)$ is a material volume containing the same moving fluid particles, then, $U_n = \mathbf{q}\cdot\mathbf{n}$ and d/dt is the material derivative. We then have as a corollary, the following

Theorem 2 *If $V(t)$ is a material volume,*

$$\frac{D}{Dt}\iiint_V G\,dV = \iiint_V \frac{\partial G}{\partial t}\,dV + \iint_S G\mathbf{q}\cdot\mathbf{n}\,dS \tag{1.3.2}$$

This is the *kinematic transport theorem.*

Let us apply this theorem to derive certain differential conservation laws.

1.3.1 Transport of Mass

Let $G = \rho =$ density and $V(t)$ be a material volume within which there are no mass sources or sinks, then by mass conservation,

$$\frac{D}{Dt}\iiint_{V(t)} \rho dV = 0$$

by mass conservation. Using (1.3.2) we get

$$\iiint_V \frac{\partial \rho}{\partial t} + \iint_S \rho \mathbf{q} \cdot \mathbf{n} dS = \iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) \right) dV = 0 \tag{1.3.3}$$

Because V is arbitrary the integrand must vanish identically :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0 \tag{1.3.4}$$

This is the differential form of mass conservation law, valid at any point inside the fluid. As an alternate form we may write

$$\frac{\partial \rho}{\partial t} + \mathbf{q} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{q} = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{q} = 0 \tag{1.3.5}$$

For the special case of an incompressible but non homogeneous fluid:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{q} \cdot \nabla \rho = 0 \tag{1.3.6}$$

It follows that

$$\nabla \cdot \mathbf{q} = 0 \tag{1.3.7}$$

which is also known as the continuity equation. If the incompressible fluid is also homogeneous, then (1.3.7) holds and

$$\rho = \text{constant} \tag{1.3.8}$$

replaces (1.3.6). Do avoid the common error of assuming incompressible fluid as being homogeneous.

1.3.2 Transport of any dynamical property

Let $G = \rho F$ where F is some fluid-dynamic property per unit mass, and V is a material volume, then,

$$\frac{D}{Dt} \iiint_V \rho F dV = \iiint_V \rho \frac{DF}{Dt} dV \tag{1.3.9}$$

Thus, the total derivative can pass through the integral sign and ρ. From (1.3.2)

$$\begin{aligned}
\frac{D}{Dt} \iiint_V \rho F dV &= \iiint_V \frac{\partial \rho F}{\partial t} dV + \iint_S \rho F \mathbf{q} \cdot \mathbf{n} dS \\
&= \iiint_V \left(\frac{\partial \rho}{\partial t} F + \rho \frac{\partial F}{\partial t} + \nabla \cdot (\rho \mathbf{q} F) \right) dV \\
&= \iiint_V \left\{ F \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) \right] + \rho \left[\frac{\partial F}{\partial t} + \mathbf{q} \cdot \nabla F \right] \right\} dV \\
&= \iiint_V \rho \frac{DF}{Dt} dV
\end{aligned}$$

after using the law of mass conservation (1.3.5).

As a special case the rate of momentum transport in a fluid volume V is found by taking $G = \rho\mathbf{q}$, then,

$$\frac{D}{Dt}\iiint_V \rho\mathbf{q}dV = \iiint_V \rho\frac{D\mathbf{q}}{Dt}dV \tag{1.3.10}$$

which will be used later.