LECTURE 2: Stress Conditions at a Fluid-fluid Interface

We proceed by deriving the normal and tangential stress boundary conditions appropriate at a fluid-fluid interface characterized by an interfacial tension σ.

Consider an interfacial surface S bound by a closed contour C (Figure 1). One may think of there being a force per unit length of magnitude σ in the s-direction at every point along C that acts to flatten the surface S. Perform a force balance on a volume element V enclosing the interfacial surface S defined by the contour C:

$$\int_V \rho \frac{D\mathbf{u}}{Dt}\, dV \;=\; \int_V \mathbf{f}\, dV \;+\; \int_S \left[\mathbf{t}(\mathbf{n}) + \hat{\mathbf{t}}(\hat{\mathbf{n}})\right] dS \;+\; \int_C \sigma \mathbf{s}\, d\ell$$

Here ℓ indicates arclength and so $d\ell$ a length increment along the curve C. $\mathbf{t}(\mathbf{n}) = \mathbf{n}\cdot\mathbf{T}$ is the stress vector, the force/area exerted by the upper (+) fluid on the interface. The stress tensor is defined in terms of the local fluid pressure and velocity field as $\mathbf{T} = -p\,\mathbf{I} + \mu[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$. Similarly, the stress exerted on the interface by the lower (-) fluid is $\hat{\mathbf{t}}(\hat{\mathbf{n}}) = \hat{\mathbf{n}}\cdot\hat{\mathbf{T}} = -\mathbf{n}\cdot\hat{\mathbf{T}}$ where $\hat{\mathbf{T}} = -\hat{p}\,\mathbf{I} + \hat{\mu}[\nabla\hat{\mathbf{u}} + (\nabla\hat{\mathbf{u}})^T]$.

Physical interpretation of terms

$\int_V \rho \frac{D\mathbf{u}}{Dt}\, dV$: inertial force associated with acceleration of fluid within V

$\int_V \mathbf{f}\, dV$: body forces acting on fluid within V

$\int_S \mathbf{t}(\mathbf{n})\, dS$: hydrodynamic force exerted at interface by fluid +

$\int_S \hat{\mathbf{t}}(\hat{\mathbf{n}})\, dS$: hydrodynamic force exerted at interface by fluid -

$\int_C \sigma\mathbf{s}\, d\ell$: surface tension force exerted along perimeter C

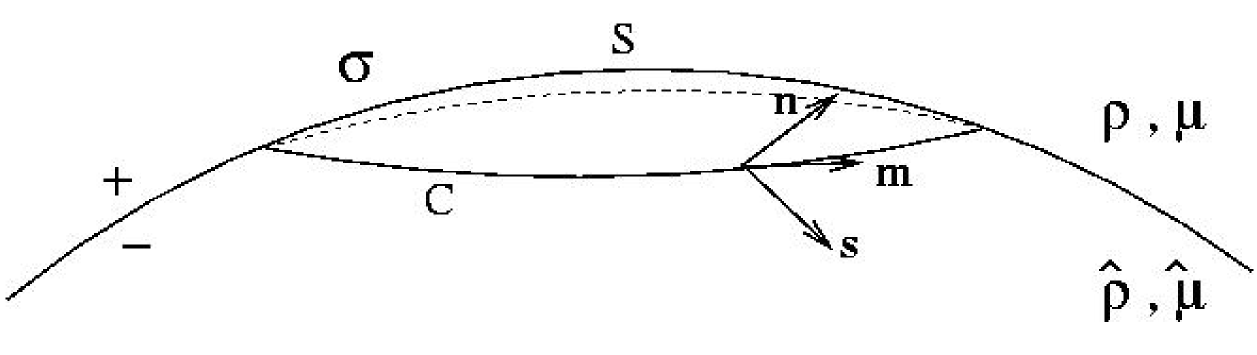

Figure 1: A surface S and bounding contour C on an interface between two fluids. The upper fluid (+) has density ρ and viscosity μ; the lower fluid (-), $\hat{\rho}$ and $\hat{\mu}$. $\mathbf{n}$ represents the unit outward normal to the surface, and $\hat{\mathbf{n}} = -\mathbf{n}$ the unit inward normal. $\mathbf{m}$ the unit tangent to the contour C and $\mathbf{s}$ the unit vector normal to C but tangent to S.

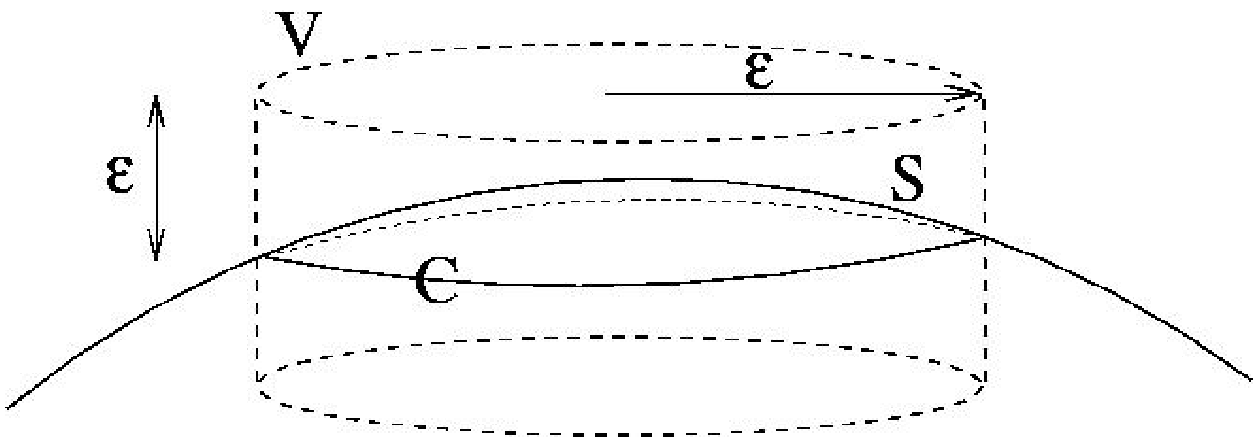

Figure 2: A Gaussian fluid pillbox of height and radius ϵ spanning the interface evolves under the combined influence of volume and surface forces.

Now if ϵ is the typical lengthscale of the element V, then the acceleration and body forces will scale as ϵ^3, but the surface forces will scale as ϵ^2. Hence, in the limit of $\epsilon \to 0$, we have that the surface forces must balance:

$$\int_S [\mathbf{t}(\mathbf{n}) + \hat{\mathbf{t}}(\hat{\mathbf{n}})]\ dS \ + \ \int_C \sigma\ \mathbf{s}\ d\ell \ = \ 0$$

Now we have that

$$\mathbf{t}(\mathbf{n}) \ = \ \mathbf{n} \cdot \mathbf{T} \qquad , \qquad \hat{\mathbf{t}}(\mathbf{n}) \ = \ \hat{\mathbf{n}} \cdot \hat{\mathbf{T}} \ = \ -\mathbf{n} \cdot \hat{\mathbf{T}}$$

Moreover, the application of Stokes Theorem (see Appendix A) allows us to write

$$\int_C \sigma \mathbf{s}\ d\ell \ = \ \int_S \nabla_s \sigma \ - \ \sigma \mathbf{n}\ (\nabla_s \cdot \mathbf{n})\ dS$$

where the tangential gradient operator, defined by

$$\nabla_s \ = \ [\mathbf{I} \ - \ \mathbf{nn}] \ \cdot \ \nabla \ = \ \nabla \ - \ \mathbf{n}\frac{\partial}{\partial n}$$

appears because σ and $\mathbf{n}$ are defined only on the surface. We proceed by dropping the subscript s on ∇, with this understanding.

The surface force balance thus becomes:

$$\int_S [\mathbf{n} \cdot \mathbf{T} \ - \ \mathbf{n} \cdot \hat{\mathbf{T}}]\ dS \ = \ \int_S \sigma \mathbf{n}\ (\nabla \cdot \mathbf{n}) \ - \ \nabla \sigma\ dS \tag{1}$$

Now since the surface element is arbitrary, the integrand must vanish identically. One thus obtains the interfacial stress balance equation.

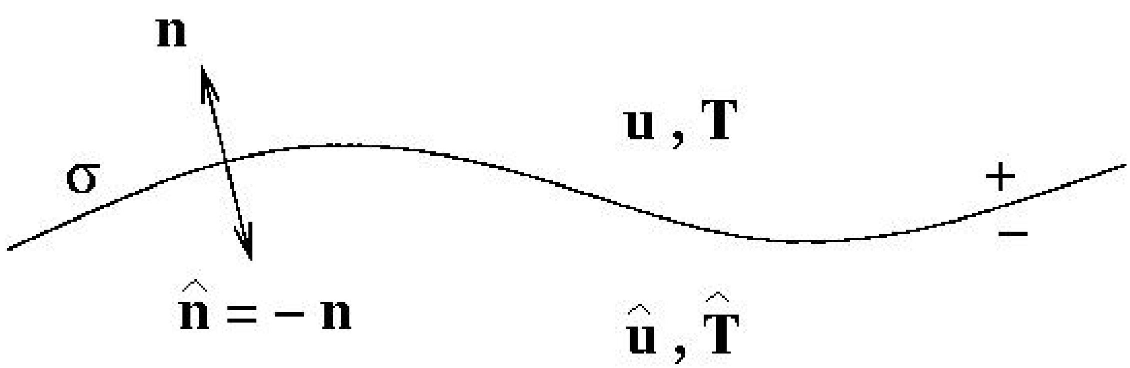

Figure 3: A definitional sketch of a fluid-fluid interface. Carats denote variables in the lower fluid.

Stress Balance Equation

$$\mathbf{n}\cdot\mathbf{T} \;-\; \mathbf{n}\cdot\hat{\mathbf{T}} \;=\; \sigma\mathbf{n}\,(\nabla\cdot\mathbf{n}) \;-\; \nabla\sigma \tag{2}$$

Interpretation of terms:

$\mathbf{n}\cdot\mathbf{T}$: stress (force/area) exerted by + on - (will generally have both normal and tangential components)

$\mathbf{n}\cdot\hat{\mathbf{T}}$: stress (force/area) exerted by - on + (will generally have both normal and tangential components)

$\sigma\mathbf{n}\,(\nabla\cdot\mathbf{n})$: normal curvature force per unit area associated with local curvature of interface, $\nabla\cdot\mathbf{n}$.

$\nabla\sigma$: tangential stress associated with gradients in surface tension.

Both normal and tangential stress must be balanced at the interface. We consider each component in turn.

Normal Stress Balance

Taking $\mathbf{n}\cdot(2)$ yields the normal stress balance at the interface:

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n} \;-\; \mathbf{n}\cdot\hat{\mathbf{T}}\cdot\mathbf{n} \;=\; \sigma\,(\nabla\cdot\mathbf{n}) \tag{3}$$

The jump in normal stress across the interface must balance the curvature force per unit area. We note that a surface with non-zero curvature ($\nabla\cdot\mathbf{n}\neq 0$) reflects a jump in normal stress across the interface.

Tangential Stress Balance

Taking $\mathbf{t}\cdot(2)$, where $\mathbf{t}$ is any unit vector tangent to the interface, yields the tangential stress balance at the interface:

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{t}\ -\ \mathbf{n}\cdot\hat{\mathbf{T}}\cdot\mathbf{t}\ =\ \nabla\sigma\cdot\mathbf{t} \tag{4}$$

Physical Interpretation:

- the LHS represents the jump in tangential components of the hydrodynamic stress at the interface
- the RHS represents the tangential stress associated with gradients in σ, as may result from gradients in temperature or chemical composition at the interface
- the LHS contains only velocity gradients, not pressure; therefore, a non-zero $\nabla\sigma$ at a fluid interface must *always* drive motion.

Appendix A

Recall Stokes Theorem:

$$\int_C \mathbf{F}\cdot\vec{d\ell}\ =\ \int_S \mathbf{n}\cdot(\nabla\wedge\mathbf{F})\ dS$$

Along the contour C, $\vec{d\ell}=\mathbf{m}\ d\ell$, so that we have

$$\int_C \mathbf{F}\cdot\mathbf{m}\ d\ell\ =\ \int_S \mathbf{n}\cdot(\nabla\wedge\mathbf{F})\ dS$$

Now let $\mathbf{F}=\mathbf{f}\wedge\mathbf{b}$, where $\mathbf{b}$ is an arbitrary *constant* vector. We thus have

$$\int_C (\mathbf{f}\wedge\mathbf{b})\cdot\mathbf{m}\ d\ell\ =\ \int_S \mathbf{n}\cdot(\nabla\wedge(\mathbf{f}\wedge\mathbf{b}))\ dS$$

Now use standard vector identities to see:

$$(\mathbf{f}\wedge\mathbf{b})\cdot\mathbf{m}\ =\ -\,\mathbf{b}\cdot(\mathbf{f}\wedge\mathbf{m})$$

$$\nabla\wedge(\mathbf{f}\wedge\mathbf{b})\ =\ \mathbf{f}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{f})+\mathbf{b}\cdot\nabla\mathbf{f}-\mathbf{f}\cdot\nabla\mathbf{b}$$

$$=\ -\,\mathbf{b}(\nabla\cdot\mathbf{f})+\mathbf{b}\cdot\nabla\mathbf{f}$$

since $\mathbf{b}$ is a constant vector. We thus have

$$\mathbf{b} \cdot \int_C (\mathbf{f} \wedge \mathbf{m})\ d\ell \;=\; \mathbf{b} \cdot \int_S \; [\mathbf{n}(\nabla \cdot \mathbf{f}) - (\nabla \mathbf{f}) \cdot \mathbf{n}]\ dS$$

Since $\mathbf{b}$ is arbitrary, we thus have

$$\int_C (\mathbf{f} \wedge \mathbf{m})\ d\ell \;=\; \int_S \; [\mathbf{n}(\nabla \cdot \mathbf{f}) - (\nabla \mathbf{f}) \cdot \mathbf{n}]\ dS$$

We now choose $\mathbf{f} = \sigma \mathbf{n}$, and recall that $\mathbf{n} \wedge \mathbf{m} = -\mathbf{s}$. One thus obtains

$$-\int_C \; \sigma \mathbf{s}\ d\ell \;=\; \int_S \; [\mathbf{n} \nabla \cdot (\sigma \mathbf{n}) \;-\; \nabla(\sigma \mathbf{n}) \cdot \mathbf{n}]\ dS$$

$$= \int_S \; [\mathbf{n} \nabla \sigma \cdot \mathbf{n} + \sigma \mathbf{n} (\nabla \cdot \mathbf{n}) - \nabla \sigma - \sigma (\nabla \mathbf{n}) \cdot \mathbf{n}\ ]\ dS$$

We note that

$\nabla \sigma \cdot \mathbf{n} = 0$ since $\nabla \sigma$ must be tangent to the surface S,

$(\nabla \mathbf{n}) \cdot \mathbf{n} \;=\; \frac{1}{2} \nabla (\mathbf{n} \cdot \mathbf{n}) \;=\; \frac{1}{2} \nabla (1) \;=\; 0$,

and so obtain the desired result:

$$\int_C \; \sigma \mathbf{s}\ d\ell \;=\; \int_S \; [\ \nabla \sigma \;-\; \sigma \mathbf{n}\ (\nabla \cdot \mathbf{n})\ ]\ dS$$